Preliminary Problem

There are three bowls of fruit sitting on a shelf so high that you can’t see into any of them. One bowl contains all apples, one bowl contains all oranges, and one bowl contains apples and oranges. Each bowl is visibly labeled with one of the labels: APPLES, ORANGES, or APPLES AND ORANGES. However, each bowl is incorrectly labeled. Your task is to select one bowl and reach in and grab one piece of fruit. Having done this and using the information above can you label each bowl correctly? Explain your answer.
Problem solving has long been recognized as one of the hallmarks of mathematics. What does problem solving mean? George Pólya (1887–1985), one of the great mathematicians and teachers of the twentieth century, pointed out that “solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable.” (Pólya 1981, p. ix)

In *Principles and Standards of School Mathematics* PSSM, (NCTM 2000), we find the following:

Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking. (p. 52)

Further, we find that

Instructional programs from pre-kindergarten through grade 12 should enable all students to

• build new mathematical knowledge through problem solving;

• solve problems that arise in mathematics and in other contexts;

• apply and adapt a variety of appropriate strategies to solve problems;

• monitor and reflect on the process of mathematical problem solving. (p. 52)

Students learn mathematics as a result of solving problems. Exercises that are routine practice for skill building serve a purpose in learning mathematics, but problem solving must be a focus of school mathematics. As pointed out in the Research Note, a reasonable amount of tension and discomfort improves problem-solving performance. Mathematical experience often determines whether situations are problems or exercises.

A reasonable amount of tension and discomfort improves students’ problem-solving performance. The motivation is the release of tension after the problem has been solved. If the tension is not present, the problem is either an exercise or the students are “generally unwilling to attack the problem in a serious way” (Bloom and Broder 1950; McLeod 1985).

Worthwhile, interesting problems, not just routine word problems, must be a part of elementary students’ mathematical experience. To engage students in worthwhile tasks, problems should be introduced in a familiar context, as seen in the cartoon below.
Good mathematical problem solving occurs when all of the following are present:

1. Students are presented with a situation that they understand but do not know how to proceed directly to a solution.
2. Students are interested in finding the solution and attempt to do so.
3. Students are required to use mathematical ideas to solve the problem.

In this text, you will have many opportunities to solve problems. Each chapter opens with a problem that can be solved by using the concepts developed in the chapter. We give a hint for the solution to the problem at the end of each chapter. Throughout the text, numerous problems are solved using a four-step process and others solved using other formats.

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**Research Note**

Students who are involved in justifying their solutions to other students, especially if there is a disagreement, will gain better mathematical understanding. Discussions of differing points of view are a valuable part of the learning experience. Mathematical language is learned in this way, as is the appreciation of the need for precision in the language (Hatano and Ingaki 1991).

As the Research Note indicates, working with other students to solve problems can enhance problem-solving ability and communication skills. We encourage cooperative learning and working in groups whenever possible. To encourage group work and help identify when cooperative learning could be useful, we identify activities that involve tasks where it might be helpful to have several people gathering data, or problems where group discussions might lead to strategies for solving the problem.

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**Historical Note**

George Pólya (1887–1985) was born in Hungary and received his Ph.D. from the University of Budapest. He moved to the United States in 1940, and after a brief stay at Brown University he joined the faculty at Stanford University. In addition to being a preeminent mathematician, he focused on the vital importance of mathematics education. At Stanford, he published 10 books, including How To Solve It (1945), which has been translated into 23 languages.
Four-Step Problem-Solving Process

1. Understanding the problem
   a. Can you state the problem in your own words?
   b. What are you trying to find or do?
   c. What are the unknowns?
   d. What information do you obtain from the problem?
   e. What information, if any, is missing or not needed?

2. Devising a plan
   The following list of strategies, although not exhaustive, is very useful:
   a. Look for a pattern.
   b. Examine related problems and determine if the same technique applied to them can be applied to the current problem.
   c. Examine a simpler or special case of the problem to gain insight into the solution of the original problem.
   d. Make a table or list.
   e. Make a diagram.
   f. Write an equation.
   g. Use guess and check.
   h. Work backward.
      i. Identify a subgoal.
   j. Use indirect reasoning.
   k. Use direct reasoning.

3. Carrying out the plan
   a. Implement the strategy or strategies in step 2 and perform any necessary actions or computations.
   b. Check each step of the plan as you proceed. This may be intuitive checking or a formal proof of each step.
   c. Keep an accurate record of your work.

4. Looking back
   a. Check the results in the original problem. (In some cases, this will require a proof.)
   b. Interpret the solution in terms of the original problem. Does your answer make sense? Is it reasonable? Does it answer the question that was asked?
   c. Determine whether there is another method of finding the solution.
   d. If possible, determine other related or more general problems for which the techniques will work.

What role should Pólya’s problem-solving process play in the teaching of elementary mathematics? This question is answered in Principles and Standards in the following way:

An obvious question is, How should these strategies be taught? Should they receive explicit attention, and how should they be integrated with the mathematics curriculum? As with any other component of the mathematical tool kit, strategies must receive instructional attention if students are expected to learn them. In the lower grades, teachers can help children express, categorize, and compare their strategies. Opportunities to use strategies must be embedded naturally in the curriculum across the content areas. By the time students reach the middle grades, they should be skilled at recognizing when various strategies are appropriate to use and should be capable of deciding when and how to use them. (p. 54)
Problem-solving ability develops slowly over time, perhaps because the many understandings and skills needed for problem solving develop at different rates. A key element in developing problem-solving skills is multiple, continuous experience in solving problems with different contexts and at different levels of ability (Kantowski 1981).

**Strategies for Problem Solving**

We next provide a variety of problems with different contexts so that you may gain experience in problem solving, as mentioned in the Research Note. Frequently, a variety of problem-solving strategies is necessary to solve these and other problems.

Strategies are tools that might be used to discover or construct the means to achieve a goal. For each strategy described next, we give an example that can be solved with that strategy. Often, problems can be solved in more than one way, as seen in the cartoon below. You may devise a different strategy to solve the sample problems. There is no one best strategy to use.

![Cartoon of a maze with multiple paths](image)

**Nontraditional Solution**

Carl Gauss (1777–1855) is regarded as the greatest mathematician of the nineteenth century and one of the greatest mathematicians of all time. Born to humble parents in Brunswick, Germany, he was an infant prodigy who, it is said, at age 3 corrected an arithmetic error in his father's bookkeeping. Gauss made contributions in the areas of astronomy, geodesy, and electricity. After Gauss's death, the King of Hanover ordered a commemorative medal prepared in his honor. On the medal was an inscription referring to Gauss as the “Prince of Mathematics,” a title that stayed with his name.
Strategy: Look for a Pattern

Problem Solving  Gauss's Problem

When Carl Gauss was a child, his teacher required the students to find the sum of the first 100 natural numbers. The teacher expected this problem to keep the class occupied for some time. Gauss gave the answer almost immediately. Can you?

Understanding the Problem  The natural numbers are 1, 2, 3, 4, . . . . Thus, the problem is to find the sum 1 + 2 + 3 + 4 + . . . + 100.

Devising a Plan  The strategy look for a pattern is useful here. One version of the story about young Gauss reports that he listed the numbers as shown in Figure 1-1.

Let Then,

\[
S = \frac{100 \cdot 101}{2},
\]

or 5050.

Looking Back  The method is mathematically correct because addition can be performed in any order, and multiplication is repeated addition. Also, the sum in each pair is always 101 because when we move from any pair to the next, we add 1 to the top and subtract 1 from the bottom, which does not change the sum; for example,

\[
2 + 99 = (1 + 1) + (100 - 1) = 1 + 100, \\
3 + 98 = (2 + 1) + (99 - 1) = 2 + 99 = 101, 
\]

and so on.

A more general problem is to find the sum of the first \( n \) natural numbers

\[
1 + 2 + 3 + 4 + 5 + 6 + \ldots + n.
\]

We use the same plan as before and notice the relationship in Figure 1-2. There are \( n \) sums of \( n + 1 \) for a total of \( n(n + 1) \). Therefore,

\[
2S = n(n + 1) \quad \text{and} \quad S = \frac{n(n + 1)}{2}. 
\]

Figure 1-2

A different strategy for finding the sum 1 + 2 + 3 + . . . + \( n \) involves the strategy of making a diagram and thinking of the sum geometrically as a stack of blocks. To find the sum, consider the stack in Figure 1-3(a) and a stack of the same size placed differently next to the original stack, as in Figure 1-3(b). The total number of blocks in the stack in Figure 1-3(b) is \( n(n + 1) \), which is twice the desired sum. Thus, the desired sum is \( n(n + 1)/2 \).
Strategy: Examine a Related Problem

Problem Solving   Sums of Even Natural Numbers

Find the sum of the even natural numbers less than or equal to 100. Devise a strategy for finding that sum and generalize the result.

Understanding the Problem   Even natural numbers are 2, 4, 6, 8, 10, . . .. The problem is to find the sum of the even natural numbers 2 + 4 + 6 + 8 + . . . + 100.

Devising a Plan   Recognizing that the sum can be separated into two simpler parts related to Gauss’s original problem helps us devise a plan. Consider the following:

\[
2 + 4 + 6 + 8 + \ldots + 100 = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + \ldots + 2 \cdot 50
\]

\[
= 2(1 + 2 + 3 + 4 + \ldots + 50)
\]

Thus, we can use Gauss’s method to find the sum of the first 50 natural numbers and then double that.
**Carrying Out the Plan**  We carry out the plan as follows:

\[
2 + 4 + 6 + 8 + \ldots + 100 = 2(1 + 2 + 3 + 4 + \ldots + 50)
= 2 \cdot [50(50 + 1)/2]
= 2550
\]

Thus, the sum is 2550.

**Looking Back**  A different way to approach this problem is to realize that there are 25 sums of 102, as shown in Figure 1-4.

\[
\begin{align*}
\text{Figure 1-4} \\
2 + 4 + 6 + 8 + \ldots + 94 + 96 + 98 + 100
\end{align*}
\]

Thus, the sum is 25 \cdot 102, or 2550.

**NOW TRY THIS 1-2**

a. Find the sum of the odd natural numbers less than 100.

b. Let \(a_1, a_2, a_3, \ldots, a_n\) be any sequence of \(n\) terms where \(a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = a_n - a_{n-1} = d\), where \(d\) is a fixed number. Write an expression for the sum of the terms in this sequence in terms of \(a_1, a_n,\) and \(n\).

**Strategy: Examine a Simpler Case**

One strategy for solving a complex problem is to examine a simpler case of the problem and then consider other parts of the complex problem. An example is shown on the student page on page 9.

**NOW TRY THIS 1-3**  Sixteen people in a round-robin handball tournament played every person once. How many games were played?

**Strategy: Make a Table**

An often-used strategy in elementary school mathematics is making a table. A table can be used to look for patterns that emerge in the problem, which in turn can lead to a solution. An example of this strategy is shown on page 10. Did Plan II really pay $128?

**NOW TRY THIS 1-4**  Molly and Karly started a new job the same day. After they start work, Molly is to visit the home office every 15 days and Karly is to visit the home office every 18 days. How many days will it be before they both visit the home office the same day?
**SCHOOL BOOK PAGE**

**SOLVING A SIMPLER PROBLEM**

**Lesson 11-8**

**Problem-Solving Strategy**

**Key Idea**

Learning how and when to solve a simpler problem can help you solve problems.

**Solve a Simpler Problem**

**LEARN**

**How do you solve a simpler problem?**

Triangle Trains: Each side of each triangle in the figure at the right is one inch. If there are 12 triangles in a row, what is the perimeter of the figure?

**Read and Understand**

**What do you know?**

Triangles are being connected. Each side of each triangle is one inch.

**What are you trying to find?**

Find the perimeter of the figure with 12 triangles.

**Plan and Solve**

**What strategy will you use?**

Strategy: Solve a Simpler Problem.

- I can look at 1 triangle, then 2 triangles, then 3 triangles.
  - perimeter = 3 inches
  - perimeter = 4 inches
  - perimeter = 5 inches

Answer: The perimeter is 2 more than the number of triangles. For 12 triangles, the perimeter is 14 inches.

**Look Back and Check**

**Is your work correct?**

Yes, I saw a correct pattern.

**Talk About It**

1. How was the problem broken apart into simpler problems?
2. Describe the pattern in the simpler problems.

Source: Scott Foresman-Addison Wesley, Grade 4, 2008 (p. 648).
Key Idea
Learning how and when to make a table can help you solve problems.

Make a Table

LEARN

How can you make and use a table to solve a problem?

Babysitting: Carrie is offered an afternoon babysitting job that will last 10 days. The parents who want to hire her offer two plans for payment. Which payment should Carrie accept?

Plan I: A single $100 payment for the 10 days worked
Plan II: Pay for the first day would be $0.25. Then each day thereafter, the total amount of pay would double.

Read and Understand

What do you know?
What are you trying to find?

There are two different plans. Find the total pay for 10 days under Plan II.

Plan and Solve

What strategy will you use?

Strategy: Make a Table

How to Make a Table

Step 1: Set up the table with the correct labels.
Step 2: Enter known data into the table.
Step 3: Look for a pattern. Extend the table.
Step 4: Find the answer in the table.

Days
Amount

1
$0.25

2
$0.50

3
$1

Days
Amount

1
$0.25

2
$0.50

3
$1

4
$2

5
$4

6
$8

7
$16

8
$32

9
$64

10
$128

Answer: Carrie should accept Plan II which pays $128.

Look Back and Check

Is your answer reasonable?
Yes, the answer should be an even number because the amounts in the table were doubled.

Source: Scott Foresman-Addison Wesley, Grade 6, 2008 (p. 156).
Strategy: Identify a Subgoal

In attempting to devise a plan for solving some problems, we may realize that the problem could be solved if the solution to a somewhat easier or more familiar related problem could be found. In such a case, finding the solution to the easier problem may become a subgoal of the primary goal of solving the original problem. The following Magic Square problem shows an example of this.

Problem Solving  A Magic Square

Arrange the numbers 1 through 9 into a square subdivided into nine smaller squares like the one shown in Figure 1-5 so that the sum of every row, column, and main diagonal is the same. (The result is a magic square.)

Understanding the Problem  We need to put each of the nine numbers 1, 2, 3, . . ., 9 in the small squares, a different number in each square, so that the sums of the numbers in each row, in each column, and in each of the two major diagonals are the same.

Devising a Plan  If we knew the fixed sum of the numbers in each row, column, and diagonal, we would have a better idea of which numbers can appear together in a single row, column, or diagonal. Thus our subgoal is to find that fixed sum. The sum of the nine numbers, 1 + 2 + 3 + . . . + 9, equals 3 times the sum in one row (why?). Consequently, the fixed sum is obtained by dividing 1 + 2 + 3 + . . . + 9, by 3. Using the process developed by Gauss, we have (1 + 2 + 3 + . . . + 9) / 3 = (9 · 10) / 2 + 3, or 45 / 3 = 15, so the sum in each row, column, and diagonal must be 15. Next, we need to decide what numbers could occupy the various squares. The number in the center space will appear in four sums, each adding to 15 (two diagonals, the second row, and the second column). Each number in the corners will appear in three sums of 15. (Do you see why?) If we write 15 as a sum of three different numbers 1 through 9 in all possible ways, we could then count how many sums contain each of the numbers 1 through 9. The numbers that appear in at least four sums are candidates for placement in the center square, whereas the numbers that appear in at least three sums are candidates for the corner squares. Thus our new subgoal is to write 15 in as many ways as possible as a sum of three different numbers from the set {1, 2, 3, . . ., 9}.

Carrying Out the Plan  The sums of 15 can be written systematically as follows:

9 + 5 + 1
9 + 4 + 2
8 + 6 + 1
8 + 5 + 2
8 + 4 + 3
7 + 6 + 2
7 + 5 + 3
6 + 5 + 4
Note that $1 + 5 + 9$ and $5 + 1 + 9$, for example, are counted as the same. Notice that 1 appears in only two sums, 2 in three sums, 3 in two sums, and so on. Table 1-1 summarizes this pattern.

<table>
<thead>
<tr>
<th>Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of sums containing the number</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

The only number that appears in four sums is 5; hence, 5 must be in the center of the square. (Why?) Because 2, 4, 6, and 8 appear 3 times each, they must go in the corners. Suppose we choose 2 for the upper left corner. Then 8 must be in the lower right corner. (Why?) This is shown in Figure 1-6(a). Now we could place 6 in the lower left corner or upper right corner. If we choose the upper right corner, we obtain the result in Figure 1-6(b). The magic square can now be completed, as shown in Figure 1-6(c).

![Figure 1-6](image_url)

**Looking Back** We have seen that 5 was the only number among the given numbers that could appear in the center. However, we had various choices for a corner, and so it seems that the magic square we found is not the only one possible. Can you find all the others?

Another way to see that 5 could be in the center square is to consider the sums $1 + 9$, $2 + 8$, $3 + 7$, $4 + 6$, as shown in Figure 1-7. We could add 5 to each to obtain 15.

![Figure 1-7](image_url)

**NOW TRY THIS 1-5** Five friends decided to give a party and split the costs equally. Al spent $4.75 on invitations, Betty spent $12 for drinks and $5.25 on vegetables, Carl spent $24 for pizza, Dani spent $6 on paper plates and napkins, and Ellen spent $13 on decorations. Determine who owes money to whom and how the money can be paid.
Strategy: Make a Diagram

It has often been said that a picture is worth a thousand words. This is particularly true in problem solving. In the following problem, *making a diagram* helps us to understand the problem and work towards a solution.

**Problem Solving 50-m Race Problem**

Bill and Jim ran a 50-m race 3 times. The speed of the runners does not vary. In the first race, Jim was at the 45-m mark when Bill crossed the finish line.

a. In the second race, to make the race closer Jim started 5 m ahead of Bill, who lined up at the starting line. Who will win the race?

b. In the third race, Jim starts at the starting line and Bill starts 5 m behind. Who will win the race?

**Understanding the Problem**  When Bill and Jim run a 50-m race, Bill wins by 5 m; that is, whenever Bill covers 50 m, at the same time Jim covers only 45 m. If Bill starts at the starting line and Jim is given a 5-m head start, we are to determine who will win the race. If Jim starts at the starting line and Bill starts 5 m behind, we are to determine who will win.

**Devising a Plan**  A strategy to determine the winner under each condition is to *make a diagram*. A diagram for the first 50-m race is given in Figure 1-8(a). In this case, Bill wins by 5 m. In the second race, Jim is given a 5-m head start and hence when Bill runs 50 m to the finish line, Jim runs only 45 m. Because Jim is 45 m from the finish line, he reaches the finish line at the same time as Bill does. This is shown in Figure 1-8(b). In the third race, because Bill starts 5 m behind, we use Figure 1-8(a) and move Bill back 5 m, as shown in Figure 1-8(c). From the diagram we can determine the results in each case.
An Introduction to Problem Solving

Carrying Out the Plan  From Figure 1-8(b) we see that if Jim is given a 5-m head start, then the race will end in a tie. If Bill starts 5 m behind Jim, then at 45 m they will be tied. Because Bill is faster than Jim, Bill will cover the last 5 m faster than Jim and win the race.

Looking Back  The diagrams show the solution makes sense and is appropriate. Other problems can be investigated involving racing and handicaps. For example, if Bill and Jim run on a 50-m oval track, how many laps will it take for Bill to lead Jim by one full lap. (Assume the same speeds as earlier.)

REMARK  In many cases, students’ solutions may involve processes that occur simultaneously: thinking through the problem and supporting that thinking by diagram making.

NOW TRY THIS 1-6  An elevator stopped at the middle floor of a building. It then moved up 4 floors, stopped, moved down 6 floors, stopped, and then moved up 10 floors and stopped. The elevator was now 3 floors from the top floor. How many floors does the building have?

Strategy: Guess and Check

In the strategy of guess and check, we first guess at a solution using as reasonable a guess as possible. Then we check to see whether the guess is correct. If not, the next step is to learn as much as possible about the solution based on the guess before making a next guess. This strategy can be regarded as a form of trial and error, where the information about the error helps us choose what trial to make next. The guess-and-check strategy is often used when a student does not know how to solve the problem more efficiently or if the student does not yet have the tools to solve the problem in a faster way. Notice on the student page on page 15, students benefit from the observed “errors,” as mentioned in the Research Note.

Students in grades 1–3 rely primarily on the guess-and-check strategy when faced with a mathematical problem. As students enter grades 6–12 this tendency decreases. Older students benefit more from the observed “errors” after a guess when formulating a new “trial” (Lester 1975).

NOW TRY THIS 1-7  A cryptarithm is a collection of words in which each unique letter represents a unique number. Find the digits that can be substituted in the following:

\[
\begin{align*}
S U N + F U N &= S W I M
\end{align*}
\]
School Book page  GUESS AND CHECK

Lesson 5-7

Problem-Solving Strategy

Try, Check, and Revise

How do you try, check, and revise?

Sale  Suzanne spent $27, not including tax, on dog supplies. She bought two of one item and one other item. What did she buy?

Read and Understand

What do you know?  She bought three items.
The two items were the same.
The prices are in the sign.
She paid $27 for all three.

What are you trying to find?  What three items did she buy?

Plan and Solve

What strategy will you use?

Strategy: Try, Check, and Revise

How to Try, Check, and Revise

Step 1  Think to make a reasonable first try.

Step 2  Check using the information given in the problem.

Step 3  Revise. Use your first try to make a reasonable second try. Check.

Step 4  Use previous tries to continue trying and checking until you get the answer.

Look Back and Check

Is your work correct?  Yes, the sum is $27 and there are two of one item and one of another item.

Source: Scott Foresman-Addison Wesley, Grade 4, 2005 (p. 278).
**Strategy: Work Backward**

In some problems, it is easier to start with the result and to work backward. This is demonstrated on the student page on page 17. Notice also that the make a diagram strategy is used.

**NOW TRY THIS 1-8** Linda has an 80 average (mean) on her 11 math tests. Her teacher tells her she can drop her single low score of 50. What is her new average?

---

**Strategy: Use Indirect Reasoning**

To show that a statement is true, it is sometimes easier to show that it is impossible for the statement to be false. We can do this by showing that if the statement were false, something contradictory or impossible would follow. This approach is useful when it is difficult to start a direct argument and when negating the given statement gives us something tangible with which to work. An example follows.

### Problem Solving Checkerboard Problem

In Figure 1-9, we are given a checkerboard with the two squares on opposite corners removed and dominoes such that each domino can cover two adjacent squares on the board. Can the dominoes be arranged in such a way that all the remaining squares on the board can be covered with no dominoes overlapping or hanging off the board? If not, why not?

---

**Understanding the Problem** Two red spaces on opposite corners were removed from the checkerboard in Figure 1-9. We are asked whether it is possible to cover the remaining 62 squares with dominoes the size of 2 squares.

**Devising a Plan** If we try to cover the board in Figure 1-9 with dominoes, we will find that the dominoes do not fit and some squares will remain uncovered. To show that there is no way to cover the board with dominoes, we use indirect reasoning. If the 62 squares in Figure 1-9 could be covered with no dominoes overlapping or hanging, it would take 31 dominoes to accomplish the task. We want to show that this implies something impossible.
Lesson 8-9
Problem-Solving Strategy

Key Idea
Learning how and when to work backward can help you solve problems.

Work Backward

How do you work backward to solve a problem?

Tunnel Vision It took workers 5 weeks to dig a 10-mile tunnel. How much had the workers completed after the first 3 weeks of digging?

Read and Understand

What do you know? The workers completed a 10-mile tunnel in 5 weeks. During week 4, they dug 2\(\frac{1}{2}\) miles. During week 5, they dug 1\(\frac{1}{4}\) miles.

What are you trying to find? How many miles of tunnel did the workers dig in the first 3 weeks?

Plan and Solve

What strategy will you use? Strategy: Work Backward

How to Work Backward
Step 1 Identify the unknown initial amount.
Step 2 Draw a picture to show each change, starting with the initial amount.
Step 3 Start at the end result. Work backward, using the inverse of each change.

Look Back and Check

Is your answer reasonable? Yes, because when I work forward from the initial amount, I get the end result.

\[6\frac{3}{4} \text{ mi} + 2\frac{1}{2} \text{ mi} + 1\frac{1}{4} \text{ mi} = 10 \text{ mi}\]

Source: Scott Foresman-Addison Wesley, Grade 5, 2008 (p. 484).
Carrying Out the Plan Each domino must cover 1 black and 1 red square. Hence, 31 dominoes would cover 31 red and 31 black squares. This is impossible, however, because the board in Figure 1-9 has 30 red and 32 black squares. Consequently, our assumption that the board in Figure 1-9 can be covered with dominoes is wrong.

Looking Back The counting of black and red squares implies that if we remove any number of squares from a checkerboard so that the number of remaining red squares differs from the number of remaining black squares, the board cannot be covered with dominoes. (Do you see why?) We could also investigate what happens when two squares of the same color are removed from an 8-by-7 board and other-sized boards. Also, is it always possible to cover the remaining board if two squares of opposite colors are removed?

NOW TRY THIS 1-9 Al, Bob, Carl, and Dan each participate in exactly one sport: swimming, baseball, basketball, or tennis. Bob plays baseball. Al can’t swim. Carl plays basketball. In what sports does each person participate?

Strategy: Use Direct Reasoning

Problem Solving Checker Games

Two people each won three games of checkers. Is it possible that only five games were played?

Solution We know that each person won three games. Reasoning directly, we see that if they each won three games and they played each other, they would have had to play six games. Thus, they could not play each other in all games and have three wins each. Could they in fact each win three games while playing only five total games and not have played each other? The answer is no and the situation is impossible.

Strategy: Write an Equation

A problem-solving strategy used in algebraic thinking is writing an equation. This strategy is very important, and it is covered in Chapter 4, “Algebraic Thinking.”

Assessment 1-1A

1. Use the approach in Gauss’s Problem to find the following sums (do not use formulas):
   a. \(1 + 2 + 3 + 4 + \ldots + 99\)
   b. \(1 + 3 + 5 + 7 + \ldots + 1001\)
2. Find the sum of \(36 + 37 + 38 + 39 + \ldots + 146 + 147\).
3. Cookies are sold singly or in packages of two or six. How many ways can you buy a dozen cookies?
4. The sign says you are leaving Missoula, Butte is 120 mi away, and Bozeman is 200 mi away. There is a rest stop halfway between Butte and Bozeman. How far is the rest stop from Missoula if both Butte and Bozeman are in the same direction?
5. Alababa, Bubba, Cory, and Dandy are in a horse race. Bubba is the slowest, Cory is faster than Alababa but slower than Dandy. Name the finishing order of the horses.
6. Frankie and Johnny began reading a novel on the same day. Frankie reads eight pages a day and Johnny reads five pages a day. If Frankie is on page 72, what page is Johnny on?
7. What is the largest sum of money—all in coins and no silver dollars—that you could have in your pocket without being able to give change for a dollar, a half-dollar, a quarter, a dime, or a nickel?

8. a. Place the digits 1, 2, 4, 5, and 7 in the following boxes so that in (i) the greatest product is obtained and in (ii) the greatest quotient is obtained:

\[
\begin{array}{c}
\times
\end{array}
\]

b. Use the same digits as in (a) to obtain (i) the least product and (ii) the least quotient.

9. Suppose you could spend $10 every minute, night and day. How much could you spend in a year? (Assume there are 365 days in a year.)

10. How many different four-digit numbers have the same digits as 1993?

11. A compass and a ruler together cost $4. The compass costs 90¢ more than the ruler. How much does the compass cost?

12. Kathy stood on the middle rung of a ladder. She climbed up three rungs, moved down five rungs, and then climbed up seven rungs. Then she climbed up the remaining six rungs to the top of the ladder. How many rungs are there in the whole ladder?

13. Same-sized cubes are glued together to form a staircase-like sequence of solids as shown:

All of the unglued faces of the cubes need to be painted. How many squares will need to be painted in (a) the 100th solid? (b) the nth solid?

14. A farmer needs to fence a rectangular piece of land. She wants the length of the field to be 80 ft longer than the width. If she has 1080 ft of fencing material, what should the length and the width of the field be?

15. One winter night the temperature fell between midnight and 5 A.M. By 9 A.M., the temperature had doubled from what it was at 5 A.M. By noon, it had risen another 10° to 32°F. What was the temperature at midnight?

16. Al, Betty, Carl, and Dan were each born in a different season. Al was born in February. Betty was not born in the fall. Carl was born in the spring. Determine which season each child was born in.

17. The 14 digits of a credit card are written in the boxes shown. If the sum of any three consecutive digits is 20, what is the value of A?

\[
\begin{array}{c}
\times
\end{array}
\]

18. a. Place the digits 4, 5, 6, 7, and 9 in the following boxes so that in (i) the greatest product is obtained and in (ii) the greatest quotient is obtained:

\[
\begin{array}{c}
\times
\end{array}
\]

b. Use the same digits as in (a) to obtain (i) the least product and (ii) the least quotient.

19. Marc goes to the store with exactly $1.00 in change. He has at least one of each coin less than a half-dollar coin, but he does not have a half-dollar coin.

a. What is the least number of coins he could have?

b. What is the greatest number of coins he could have?

20. Find a 3-by-3 magic square using the numbers 3, 5, 7, 9, 11, 13, 15, 17, and 19.
9. Eight marbles look alike, but one is slightly heavier than the others. Using a balance scale, explain how you can determine the heavier one in exactly
   a. three weighings
   b. two weighings

10. a. Find the sum of all the numbers in the following array:
    
    \[
    \begin{array}{cccccccc}
    1 & 2 & 3 & 4 & 5 & 6 & \ldots & 100 \\
    2 & 4 & 6 & 8 & 10 & 12 & \ldots & 200 \\
    3 & 6 & 9 & 12 & 15 & 18 & \ldots & 300 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    100 & 200 & 300 & 400 & 500 & 600 & \ldots & 100 \cdot 100 \\
    \end{array}
    \]

  *b. Generalize part (a) to a similar array in which each row has \( n \) numbers and there are \( n \) rows.

11. Recall the song “The Twelve Days of Christmas”:

   \[
   \begin{align*}
   \text{On the first day of Christmas my true love gave to me a partridge in a pear tree.} \\
   \text{On the second day of Christmas my true love gave to me two turtle doves and a partridge in a pear tree.} \\
   \text{On the third day of Christmas my true love gave to me three French hens, two turtle doves, and a partridge in a pear tree.} \\
   \end{align*}
   \]

   This pattern continues for 9 more days. After 12 days,
   a. which gifts did my true love give the most? (Yes, you will have to remember the song.)
   b. how many total gifts did my true love give to me?

12. a. Using the existing lines on the checkerboard shown, how many squares are there?

   ![Checkerboard](image)

   b. If the number of rows and columns of the checkerboard is doubled, is the number of squares doubled? Justify your answer.

13. Suppose you throw three darts at the target pictured below. All of the darts hit the target for a score. What are all the different possible scores?

   ![Target](image)

14. The following is a magic square (all rows, columns, and diagonals sum to the same number). Find the value of each variable.

   \[
   \begin{array}{ccc}
   & & a \\
   17 & b & 7 \\
   12 & 22 & \ \\
   c & d & 27 \\
   \end{array}
   \]

15. Two cards are on a table. A 12 is written on one of the cards and 9 on the other. Each card has a number written on the flip side. By turning over one card, both cards, or neither card and adding the two numbers, you can get the sums of 15, 16, 20, and 21. What number is written on the flip side of each card?

16. Suppose you buy lunch for the math club. You have enough money to buy 20 salads or 15 sandwiches. The group wants 12 sandwiches. How many salads can you buy?

17. a. Suppose you have quarters, dimes, and pennies with a total value of $1.19. How many of each coin can you have without being able to make change for a dollar?

   b. Tell why the combination of coins you have in part (a) is the greatest amount of money that you can have without being able to make change for a dollar.

18. You have two containers. One holds 7 cups and one holds 4 cups. How can you measure exactly 5 cups of water if you have an unlimited amount of water to start with?
Open-Ended

4. Use exactly four 4s and any mathematical symbols to create the natural numbers 1 to 20 inclusive; for example, $4/4 + 4/4 = 2$ and $4 \times 4 + 4 - \sqrt{4} = 18$.

5. Choose a problem-solving strategy and make up a problem that would use this strategy. Write the solution using Pólya’s four-step approach.

Cooperative Learning

6. Have each person in your group work the following problem: If eight people shake hands with one another, how many handshakes take place?
   a. Compare your strategies for working the problem. How are they the same? How are they different?
   b. Find as many ways as possible to do the problem.
   c. Generalize the solution for $n$ people.

7. The distance around the world is approximately 40,000 km. Approximately how many people of average size in your group would it take to stretch around the world if they were holding hands?

8. Work in pairs on the following version of a game called NIM. A calculator is needed for each pair.
   a. Player 1 presses $[1]$ and $+$ or $[2]$ and $+$. Player 2 does the same. The players take turns until the target number of 21 is reached. The first player to make the display read 21 is the winner. Determine a strategy for deciding who always wins.
   b. Try a game of NIM using the digits 1, 2, 3, and 4, with a target number of 104. The first player to reach 104 wins. What is the winning strategy?
   c. Try a game of NIM using the digits 3, 5, and 7, with a target number of 73. The first player to exceed 73 loses. What is the winning strategy?
   d. Now play Reverse NIM with the keys $[1]$ and $[2]$. Instead of $+$, use $-$. Put 21 on the display. Let the target number be 0. Determine a strategy for winning Reverse NIM.
   e. Try Reverse NIM using the digits 1, 2, and 3 starting with 24 on the display. The target number is 0. What is the winning strategy?
   f. Try Reverse NIM using the digits 3, 5, and 7 and starting with 73 on the display. The first player to display a negative number loses. What is the winning strategy?

9. When a book is printed, pages are passed through a printing press and then folded to form a book. To see how this works, start out with a simple book made from one $8\frac{1}{2} \times 11$ in. sheet of paper. Fold the page in half lengthwise to form a book and number the pages 1 through 4. When you open your sheet of paper, the numbers 2 and 3 are on one side of the paper and the numbers 1 and 4 are on the other side. The sum of the numbers on each side of the sheet equals 5, and the sum of all the page numbers is 10. If two sheets of paper are used similarly to build an eight-page book and the pages are numbered, predict the sum of the numbers on each page and the sum of all of the page numbers. Build your book to see if you were correct. Try the same thing with three pages.
   a. Suppose you are building a 100-page book. How many sheets will you need?
   b. What is the sum of the two page numbers that occur on the same side of a sheet?
   c. What is the sum of all the page numbers in this book?
   d. Suppose you have $n$ sheets of paper. Generalize to find the number of book pages, the sum of the numbers on the same side of the sheet, and the sum of all the page numbers.

Questions from the Classroom

10. Amy asks what “problem solving” is and whether is a problem? What do you tell her?
11. John asks why the last step of Pólya’s four-step problem-solving process, looking back, is necessary since he has already given the answer. What could you tell him?
12. A student asks why he just can’t make “random guesses” rather than “intelligent guesses” when using the guess-and-check problem-solving strategy. How do you respond?
13. Rob says that it is possible to create a magic square with the numbers 1, 3, 4, 5, 6, 7, 8, 9, and 10. How do you respond?

Third International Mathematics and Science Study (TIMSS) Question

<table>
<thead>
<tr>
<th>4</th>
<th>11</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

The rule for the table is that numbers in each row and column must add up to the same number. What number goes in the center of the table?
   a. 1  
   b. 2  
   c. 7  
   d. 12

TIMSS 2003, Grade 4

National Assessment of Educational Progress (NAEP) Question

There will be 58 people at a breakfast and each person will eat 2 eggs. There are 12 eggs in each carton. How many cartons of eggs will be needed for the breakfast?
   a. 9  
   b. 10  
   c. 72  
   d. 116

NAEP 2007, Grade 4
BRAIN TEASER Ten women are fishing all in a row in a boat. One seat in the center of the boat is empty. The five women in the front of the boat want to change seats with the five women in the back of the boat. A person can move from her seat to the next empty seat or she can step over one person without capsizing the boat. What is the minimum number of moves needed for the five women in front to change places with the five in back?

LABORATORY ACTIVITY Place a half-dollar, a quarter, and a nickel in position A as shown in Figure 1-10. Try to move these coins, one at a time, to position C. At no time may a larger coin be placed on a smaller coin. Coins may be placed in position B. How many moves does it take to get them to position C? Now add a penny to the pile and see how many moves are required. This is a simple case of the famous Tower of Hanoi problem, in which ancient Brahman priests were required to move a pile of 64 disks of decreasing size, after which the world would end. How long would it take at a rate of one move per second?

1-2 Explorations with Patterns

Mathematics has been described as the study of patterns. Patterns are everywhere—in wallpaper, tiles, traffic, and even television schedules. Police investigators study case files to find the modus operandi, or pattern of operation, when a series of crimes is committed. Scientists look for patterns in order to isolate variables so that they can reach valid conclusions in their research. In Principles and Standards, we find the following:

\[\ldots\] students should investigate numerical and geometric patterns and express them mathematically in words or in symbols. They should analyze the structure of the pattern and how it grows or changes, organize this information systematically, and use their analysis to develop generalizations about the mathematical relationships in the pattern. (p. 159)

Patterns do not always have to be numerical, as shown in Now Try This 1-10.
Patterns can be surprising. Consider Example 1-1.

Example 1-1

a. Describe any patterns seen in the following:

\[
\begin{align*}
1 + 0 \cdot 9 &= 1, \\
2 + 1 \cdot 9 &= 11, \\
3 + 12 \cdot 9 &= 111, \\
4 + 123 \cdot 9 &= 1111, \\
5 + 1234 \cdot 9 &= 11111
\end{align*}
\]

b. Do the patterns continue? Why or why not?

Solution

a. There are several possible patterns. For example, the numbers on the far left are natural numbers, that is, numbers from the set \{1, 2, 3, 4, 5, \ldots\}. The pattern starts with 1 and continues to the next greater natural number in each successive line. The numbers “in the middle” are products of two numbers, the second of which is 9. The first number in the first product is 0; after that the first number is formed using natural numbers and including one more in each successive line. The resulting numbers on the right are formed using 1s and include an additional 1 in each successive line.

b. The pattern in the complete equation appears to continue for a number of cases, but it does not continue in general; for example,

\[
13 + 12345678910111213 \cdot 9 = 1,111,111,101,910,021
\]

This pattern breaks down when the pattern of digits in the number being multiplied by 9 contains previously used digits.

As seen in Example 1-1, determining a pattern on the basis of a few cases is not reliable. For all patterns found, we should either find a counterexample to show the pattern does not hold in general or justify that the pattern always works.

In Principles and Standards we find the following:

When students make a discovery or determine a fact, rather than tell them whether it holds for all numbers or if it is correct, the teacher should help the students make that determination themselves. Teachers should ask such questions as “How do you know it is true?” and should also model ways that students can verify or disprove their conjectures. In this way, students gradually develop the abilities to determine whether an assertion is true, a generalization valid, or an answer correct and to do it on their own instead of depending on the authority of the teacher or the book. (p. 126)
Inductive Reasoning

Scientists make observations and propose general laws based on patterns. Statisticians use patterns when they form conclusions based on collected data. This process, **inductive reasoning**, is the method of making generalizations based on observations and patterns. Although inductive reasoning may lead to new discoveries, its weakness is that conclusions are drawn only from the collected evidence. If not all cases have been checked, another case may prove the conclusion false. In mathematics, inductive reasoning may lead to a **conjecture**, a statement thought to be true but not yet proved true or false. For example, considering only that and that , we might conjecture that every number squared is equal to itself.

When we find an example that contradicts the conjecture, we provide a **counterexample** and prove the conjecture false in general. Students have trouble with the concept of a counterexample, as pointed out in the Research Note. To show that the preceding conjecture is false, we need exhibit only one counterexample, for example, . Sometimes finding a counterexample is difficult, and not finding one right away does not make a conjecture true.

Next, consider a pattern that does work and helps solve a problem. How can you find the sum of three consecutive natural numbers without performing the addition? Several examples are given here. Look for a pattern in these examples.

After studying the sums, you see a pattern of multiplying the middle number by 3. You could try other numbers to see if a counterexample can be found. The pattern suggests other mathematical questions you could consider. For example,

1. Does this work for any three consecutive natural numbers?
2. How can you find the sum of any odd number of consecutive natural numbers?
3. What happens if there is an even number of consecutive natural numbers?

To answer question (1), we give a proof showing that the sum of three consecutive natural numbers is equal to 3 times the middle number.

**Proof**

Let be the first of three consecutive natural numbers. Then the three numbers are , , and . The sum of these three numbers is . Therefore, the sum of the three consecutive natural numbers is times the middle number.

The Danger of Making Conjectures Based on a Few Cases

In **Principle and Standards** we find the following:

> During grades 3–5, students should move toward reasoning that depends on relationships and properties. Students need to be challenged with questions such as, What if I gave you twenty more problems like this to do—would they all work the same way? How do you know? (p. 190)

This concept is further emphasized in the Research Note.
The following discussion illustrates the danger of making a conjecture based on a few cases. In Figure 1-11, we choose points on a circle and connect them to form distinct, nonoverlapping regions. In this figure, 2 points determine 2 regions, 3 points determine 4 regions, and 4 points determine 8 regions. What is the maximum number of regions that would be determined by 10 points?

![Figure 1-11](image)

The data from Figure 1-11 are recorded in Table 1-2. It appears that each time we increase the number of points by 1, we double the number of regions. If this were true, then for 5 points we would have 2 times the number of regions with 4 points, or $2 \cdot 8 = 16 = 2^4$, and so on. If we base our conjecture on this pattern, we might believe that for 10 points, we would have $2^9$, or 512 regions. (Why?)

![Table 1-2](image)

An initial check for this conjecture is to see whether we obtain 16 regions for 5 points. We obtain a diagram similar to that in Figure 1-12, and our guess of 16 regions is confirmed. For 6 points, the pattern predicts that the number of regions will be 32. Choose the points so that they are neither symmetrically arranged nor equally spaced and count the regions carefully. You should obtain 31 regions and not 32 regions as predicted. No matter how the points are located on the circle, the guess of 32 regions is not correct. The counterexample tells us that the doubling pattern is not correct; note that it does not tell us whether or not there are 512 regions with 10 points, but only that the pattern is not what we conjectured.

A natural-looking pattern of 2, 4, 8, 16, … is suggested in this example, but the pattern does not continue, as shown when actual pictures are drawn. If we look only at the first four terms of the sequence 2, 4, 8, 16 without context, the doubling pattern is logical. In the context of counting the number of regions of a circle, however, the pattern is incorrect.
Arithmetic Sequences

A sequence is an ordered arrangement of numbers, figures, or objects. A sequence has items or terms identified as 1st, 2nd, 3rd, and so on. Often, sequences can be classified by their properties. For example, what property do the following first three sequences have that the fourth does not?

a. 1, 2, 3, 4, 5, 6
b. 0, 5, 10, 15, 20, 25
c. 2, 6, 10, 14, 18, 22
d. 1, 11, 111, 1111, 11111, 111111

In each of the first three sequences, each term—starting from the second—is obtained from the preceding one by adding a fixed number called the common difference or difference. In part (a) the difference is 1, in part (b) the difference is 5, and in part (c) the difference is 4. Sequences such as the first three are arithmetic sequences. An arithmetic sequence is one in which each successive term from the second term on is obtained from the previous term by the addition or subtraction of a fixed number. The sequence in part (d) is not arithmetic because there is no single fixed number you can add to or subtract from the previous term to obtain the next term.

Arithmetic sequences can be generated from objects, as shown in Example 1-2.

Example 1-2

Find a pattern in the number of matchsticks required to continue the pattern shown in Figure 1-13.

Solution  Assume the matchsticks are arranged so that each figure has one more square on the right than the preceding figure. Note that the addition of a square to an arrangement requires the addition of three matchsticks each time. Thus, the numerical pattern obtained is 4, 7, 10, 13, 16, 19, . . . , an arithmetic sequence with a difference of 3.
An informal description of an arithmetic sequence is one that can be described as an “add $d$” pattern, where $d$ is the common difference. In Example 1-2, $d = 3$. In the language of children, the pattern in Example 1-2 is “add 3.” This is an example of a recursive pattern. In a recursive pattern, after one or more consecutive terms are given to start, each successive term of the sequence is obtained from the previous term(s). For example, 3, 6, 9, ... is another “add 3” sequence starting with 3, and 1, 2, 3, 5, 8, 13, ... is a recursive pattern in which the next term starting with the third is obtained by adding the two previous terms.

A recursive pattern is typically used in a spreadsheet, as seen in Table 1-4 where column A tracks the order of the terms; the headers for the columns are A, B, etc. The first entry in the B column (in the B1 cell) is 4; and to find the term in the B2 cell, we use the number in the B1 cell and add 3. Once we find the B2 cell entry, the pattern is continued using the Fill Down command. In spreadsheet language, the formula $=B1 + 3$ finds any term after the first by adding 3 to the previous term. The formula, based on a recursive pattern, is a recursive formula. (For more explicit directions on using a spreadsheet, see the Technology Manual.)

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
</tr>
<tr>
<td>7</td>
<td>22</td>
</tr>
<tr>
<td>8</td>
<td>25</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

If you want to find the number of matchsticks in the 100th figure in Example 1-2, you can use the spreadsheet or you can find an explicit formula or a general rule for finding the number of matchsticks when given the number of the term. The problem-solving strategy of making a table is again helpful here.

The spreadsheet in Table 1-4 provides an easy way to make a table. Column A gives the numbers of the terms and column B gives the terms of the sequence. If you are building such a table without a spreadsheet, it might look like Table 1-5. An ellipsis, denoted by three dots, indicates that the sequence continues in the same manner. Notice that each term is a sum of 4 and a certain number of 3s. We see that the number of 3s is 1 less than the number of the term. This pattern should continue, since the first term is $4 + 0\cdot 3$ and each time we increase the number of the term by 1, we add one more 3. Thus, it seems that the 100th term is $4 + (100 - 1)3$, and, in general, the $n$th term, $a_n$, is $4 + (n - 1)3$. We write this as $a_n = 4 + (n - 1)3$. Note that $4 + (n - 1)3$ could be written as $3n + 1$. 

---

An Introduction to Problem Solving

Table 1-5

<table>
<thead>
<tr>
<th>Number of Term</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>$7 = 4 + 3 = 4 \cdot 1 + 3$</td>
</tr>
<tr>
<td>3</td>
<td>$10 = (4 + 1 \cdot 3) + 3 = 4 + 2 \cdot 3$</td>
</tr>
<tr>
<td>4</td>
<td>$13 = (4 + 2 \cdot 3) + 3 = 4 + 3 \cdot 3$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$4 + (n - 1)3$</td>
</tr>
</tbody>
</table>

Still a different approach to finding the number of matchsticks in the 100th term of Figure 1-13 might be as follows: If the matchstick figure has 100 squares, we could find the total number of matchsticks by adding the number of horizontal and vertical sticks. There are $2 \cdot 100$ placed horizontally. (Why?) Notice that in the first figure, there are 2 matchsticks placed vertically; in the second, 3; and in the third, 4. In the 100th figure, there should be $100 + 1$ vertical matchsticks. Altogether there will be $2 \cdot 100 + (100 + 1)$, or 301, matchsticks in the 100th figure. Similarly, in the $n$th figure, there would be $2n$ horizontal and $(n + 1)$ vertical matchsticks, for a total of $3n + 1$. This discussion is summarized in Table 1-6.

Table 1-6

<table>
<thead>
<tr>
<th>Number of Term</th>
<th>Number of Matchsticks Horizontally</th>
<th>Number of Matchsticks Vertically</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>200</td>
<td>101</td>
<td>301</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$2n$</td>
<td>$n + 1$</td>
<td>$2n + (n + 1) = 3n + 1$</td>
</tr>
</tbody>
</table>

If we are given the value of the term, we can use the formula for the $n$th term in Table 1-6 to work backward to find the number of the term. For example, given the term 1798, we know that $3n + 1 = 1798$. Therefore, $3n = 1797$ and $n = 599$. Consequently, 1798 is the 599th term. We could obtain the same answer by solving $4 + (n - 1)3 = 1798$ for $n$.

In the matchstick problem, we found the $n$th term of a sequence. If the $n$th term of a sequence is given, we can find any term of the sequence, as shown in Example 1-3.

Example 1-3

Find the first four terms of a sequence the $n$th term of which is given by the following and determine whether the sequence seems to be arithmetic:

a. $a_n = 4n + 3$

b. $a_n = n^2 - 1$
**Solution a.**

<table>
<thead>
<tr>
<th>Number of Term</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4 \cdot 1 + 3 = 7$</td>
</tr>
<tr>
<td>2</td>
<td>$4 \cdot 2 + 3 = 11$</td>
</tr>
<tr>
<td>3</td>
<td>$4 \cdot 3 + 3 = 15$</td>
</tr>
<tr>
<td>4</td>
<td>$4 \cdot 4 + 3 = 19$</td>
</tr>
</tbody>
</table>

Hence, the first four terms of the sequence are 7, 11, 15, 19. This sequence seems arithmetic with difference 4.

**Solution b.**

<table>
<thead>
<tr>
<th>Number of Term</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1^2 - 1 = 0$</td>
</tr>
<tr>
<td>2</td>
<td>$2^2 - 1 = 3$</td>
</tr>
<tr>
<td>3</td>
<td>$3^2 - 1 = 8$</td>
</tr>
<tr>
<td>4</td>
<td>$4^2 - 1 = 15$</td>
</tr>
</tbody>
</table>

Thus, the first four terms of the sequence are 0, 3, 8, 15. This sequence is not arithmetic because it has no common difference.

**Generalizing Arithmetic Sequences**

To generalize our work with arithmetic sequences, suppose the first term in an arithmetic sequence is $a_1$ and the difference is $d$. The strategy of making a table can be used to investigate the general term for the sequence $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots$ as shown in Table 1-7.

The $n$th term of any sequence with first term $a_1$ and difference $d$ is given by $a_n = a_1 + (n - 1)d$. For example, in the arithmetic sequence $5, 9, 13, 17, 21, 25, \ldots$, the first term is 5 and the difference is 4. Thus, the $n$th term is given by $a_1 + (n - 1)d = 5 + (n - 1)4$. Simplifying algebraically, we obtain $5 + (n - 1)4 = 5 + 4n - 4 = 4n + 1$. Check to see if $4n + 1$ generates the sequence $5, 9, 13, 17, 21, \ldots$.

**Table 1-7**

<table>
<thead>
<tr>
<th>Number of Term</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1$</td>
</tr>
<tr>
<td>2</td>
<td>$a_1 + d$</td>
</tr>
<tr>
<td>3</td>
<td>$a_1 + 2d$</td>
</tr>
<tr>
<td>4</td>
<td>$a_1 + 3d$</td>
</tr>
<tr>
<td>5</td>
<td>$a_1 + 4d$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$n$</td>
<td>$a_1 + (n - 1)d$</td>
</tr>
</tbody>
</table>

**Remark** The $n$th term of any arithmetic sequence with first term $a_1$ and difference $d$ is $a_n = a_1 + (n - 1)d$, where $n$ is a natural number but there are no restrictions on $d$. 

**NOW TRY THIS 1-12** In an arithmetic sequence with the 2nd term 11 and the 5th term 23, find the 100th term.
Example 1-4

The diagrams in Figure 1-14 show the molecular structure of alkanes, a class of hydrocarbons. C represents a carbon atom and H a hydrogen atom. A connecting segment shows a chemical bond. (Remark: \( \text{CH}_4 \) stands for \( \text{C}_1\text{H}_4 \).)

![Diagram of alkanes with molecular structures]

- **methane** (\( \text{C}_1\text{H}_4 \))
- **ethane** (\( \text{C}_2\text{H}_6 \))
- **propane** (\( \text{C}_3\text{H}_8 \))

Figure 1-14

a. Hectane is an alkane with 100 carbon atoms. How many hydrogen atoms does it have?
b. Write a general rule for alkanes \( \text{C}_n\text{H}_m \) showing the relationship between \( m \) and \( n \).

**Solution**

a. To determine the relationship between the number of carbon and hydrogen atoms, we first study the drawing of the alkanes and disregard the extreme left and right hydrogen atoms in each. With this restriction, we see that for every carbon atom, there are two hydrogen atoms. Therefore, there are twice as many hydrogen atoms as carbon atoms plus the two hydrogen atoms at the extremes. For example, when there are 3 carbon atoms, there are \((2 \cdot 3) + 2\), or 8, hydrogen atoms. This notion is summarized in Table 1-8. If we extend the table for 4 carbon atoms, we get \((2 \cdot 4) + 2\), or 10, hydrogen atoms. For 100 carbon atoms, there are \((2 \cdot 100) + 2\), or 202, hydrogen atoms.

b. In general, for \( n \) carbon atoms there would be \( n \) hydrogen atoms attached above, \( n \) attached below, and 2 attached on the sides. Hence, the total number of hydrogen atoms would be \( 2n + 2 \). Because the number of hydrogen atoms was designated by \( m \), it follows that \( m = 2n + 2 \).

<table>
<thead>
<tr>
<th>No. of Carbon Atoms</th>
<th>No. of Hydrogen Atoms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>( n )</td>
<td>( m )</td>
</tr>
</tbody>
</table>

Example 1-5

A theater is set up in such a way that there are 20 seats in the first row and 4 additional seats in each consecutive row. The last row has 144 seats. How many rows are there in the theater?

**Solution**

Because there are 4 additional seats in each consecutive row, the numbers of seats in the rows forms an arithmetic sequence. The first term, \( a_1 \), of the sequence is 20 and the
difference, \(d\), is 4. The last term in the sequence is 144. A computerized spreadsheet could easily be used to count the number of terms in the sequence, 20, 24, 28, \ldots, 144. However, without technology, we could find the number of the terms as follows: In an arithmetic sequence, \(a_n = a_1 + (n - 1)d\), where \(a_1\) is the first term, \(d\) is the difference, and \(n\) is the number of the term. In this case, \(a_1 = 20\) and \(d = 4\). Therefore,

\[
a_n = a_1 + (n - 1)d = 20 + (n - 1)4
\]

We now want to find the number of the term when \(a_n = 20 + (n - 1)4\) is equal to 144. Therefore,

\[
20 + (n - 1)4 = 144
\]

\[
(n - 1)4 = 124
\]

\[
n - 1 = 31
\]

\[
n = 32
\]

This shows that there are 32 rows in the theater.

### Fibonacci Sequence

The popular book *The Da Vinci Code* brought renewed interest to one of the most famous sequences of all time, the **Fibonacci sequence**. The Fibonacci sequence is hinted at in the following cartoon. Can you figure out a rule for the Fibonacci Sequence?

![Fibonacci cartoon](image)

**Historical Note**

Leonardo de Pisa was born in Italy around 1170. His real family name was Bonaccio but he preferred the nickname Fibonacci, derived from the Latin for *filius Bonacci*, meaning “son of Bonacci.” In his travels, Leonardo learned the Hindu-Arabic number system from the Moors. In his book *Liber Abaci* (1202) he described the workings of the Hindu-Arabic system. One of the problems in his book was the now-famous rabbit problem, whose solution is the sequence 1, 1, 2, 3, 5, 8, 13, 21, \ldots, which became known as the **Fibonacci sequence**.
The Fibonacci sequence in the cartoon had 0 as a starting term. More typically, the sequence is seen as follows:

\[ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots \]

The sequence is named after the Italian Leonardo de Pisa, better known by the nickname Fibonacci. This sequence is not arithmetic as there is no fixed difference, \( d \).

The standard mathematical way to represent a Fibonacci number has \( F_1 \) representing the first term, \( F_2 \) representing the second term, \( F_3 \) representing the third term, and in general \( F_n \) representing the \( n \)th term. If we want to indicate the Fibonacci numbers that come after \( F_n \), we write them as \( F_{n+1}, F_{n+2}, \) and so on. The number that comes before \( F_n \) is \( F_{n-1} \). With this notation, the rule for generating the Fibonacci sequence can be written as

\[ F_n = F_{n-1} + F_{n-2}, \quad \text{for } n = 3, 4, 5, \ldots \]

Notice that this rule cannot be applied to the first two Fibonacci numbers. Because \( F_1 = 1 \) and \( F_2 = 1 \), then \( F_3 = 1 + 1 = 2 \). The seeds \( F_1 = 1 \) and \( F_2 = 1 \) and the rule \( F_n = F_{n-1} + F_{n-2} \) give another example of a recursive definition because the rule in the sequence defines a number using previous numbers in the same sequence. Using the seeds and the rule, we can find any Fibonacci number. To find \( F_{100} \) with what we know right now, we would have to know \( F_{98} \) and \( F_{99} \). A spreadsheet can easily generate this sequence.

**NOW TRY THIS 1-13**

- a. Add the first three Fibonacci numbers.
- b. Add the first four Fibonacci numbers.
- c. Add the first five Fibonacci numbers.
- d. Add the first six Fibonacci numbers.
- e. Add the first seven Fibonacci numbers.
- f. What pattern is there in the sums in parts (a)–(e) and any of the remaining numbers in the Fibonacci sequence?
- g. Write a rule for your pattern in part (f) using the notation for Fibonacci numbers.

**Geometric Sequences**

A child has 2 biological parents, 4 grandparents, 8 great-grandparents, 16 great-great-grandparents, and so on. The number of generational ancestors form the geometric sequence \( 2, 4, 8, 16, 32, \ldots \). Each successive term of a geometric sequence is obtained from its predecessor by multiplying by a fixed nonzero number, the ratio. In this example, both the first term and the ratio are 2. (The ratio is 2 because each person has two parents.) To find the \( n \)th term, \( a_n \), examine the pattern in Table 1-9.

In Table 1-9, when the given term is written as a power of 2, the number of the term is the exponent. Following this pattern, the 10th term, \( a_{10} \), is \( 2^{10} \), or 1024. The 100th term,
Section 1-2  Explorations with Patterns  33

$\alpha_{100}$ is $2^{100}$, and the $n$th term, $\alpha_n$, is $2^n$. Thus, the number of ancestors in the $n$th previous generation is $2^n$. The notation used in Table 1-9 can be generalized as follows.

| Table 1-9 |
|---|---|
| **Number of Term** | **Term** |
| 1 | $2 = 2^1$ |
| 2 | $4 = 2 \cdot 2 = 2^2$ |
| 3 | $8 = (2 \cdot 2) \cdot 2 = 2^3$ |
| 4 | $16 = (2 \cdot 2 \cdot 2) \cdot 2 = 2^4$ |
| 5 | $32 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot 2 = 2^5$ |
| $\cdot$ | $\cdot$ |
| $\cdot$ | $\cdot$ |

**Definition**

If $n$ is a natural number, then $a^n = a \cdot a \cdot a \cdot \ldots \cdot a$. If $n = 0$ and $a \neq 0$, then $a^0 = 1$.

Geometric sequences play an important role in everyday life. For example, suppose you have $1000 in a bank that pays 5% interest annually. If no money is added or taken out, then at the end of the first year you have all of the money you started with plus 5% more, that is,

**Year 1:** $1000 + 0.05(1000) = 1000(1 + 0.05) = 1000(1.05) = 1050$

If no money is added or taken out, then at the end of the second year you would have 5% more money than the previous year.

**Year 2:** $1050 + 0.05(1050) = 1050(1 + 0.05) = 1050(1.05) = 1102.50$

The amount of money in the account after any number of years can be found by noting that every dollar invested for one year becomes $1 + 0.05 \cdot 1$, or 1.05 dollars. Therefore, the amount in each year is obtained by multiplying the amount from the previous year by 1.05. The amounts in the bank after each year form a geometric sequence because the amount in each year (starting from year 2) is obtained by multiplying the amount in the previous year by the same number, 1.05. This is summarized in Table 1-10.

| Table 1-10 |
|---|---|
| **Number of Term (Year)** | **Term (Amount at the beginning of each year)** |
| 1 | $1000$ |
| 2 | $1000(1.05)$
| 3 | $1000(1.05)^2 = 1102.50$ |
| 4 | $1000(1.05)^3 = 1157.63$ |
| $\cdot$ | $\cdot$ |
| $\cdot$ | $\cdot$ |
| $\cdot$ | $\cdot$ |
| $n$ | $1000(1.05)^{n-1}$ |
Finding the $n$th Term for a Geometric Sequence

It is possible to find the $n$th term, $a_n$, of any geometric sequence when given the first term and the ratio. If the first term is $a_1$ and the ratio is $r$, then the terms are as listed in Table 1-11. Notice that the second term is $a_1r$, the third term is $a_1r^2$, and the fourth term is $a_1r^3$. The power of $r$ in each term is 1 less than the number of the term. This pattern continues since we multiply by $r$ to get the next term. Thus, the $n$th term, $a_n$, is $a_1r^{n-1}$. For $n = 1$, we have $a_1r^{1-1} = a_1r^0$. Because the first term is $a_1$, $a_1r^0 = a_1$. For all numbers $r \neq 0$, we define $r^0 = 1$. For the geometric sequence 3, 12, 48, 192, ..., the first term is 3 and the ratio is 4, and so the $n$th term, $a_n$, is given by $a_n = a_1r^{n-1} = 3 \cdot 4^{n-1}$.

**Remark** The $n$th term of any geometric sequence with first term $a_1$ and a ratio $r$ is $a_n = a_1 \cdot r^{n-1}$, when $n$ is a natural number and $r \neq 0$.

**NOW TRY THIS 1-14**

a. Two bacteria are in a dish. The number of bacteria triples every hour. Following this pattern, find the number of bacteria in the dish after 10 hours and after $n$ hours.

b. Suppose that instead of increasing geometrically as in part (a), the number of bacteria increases arithmetically by 3 each hour. Compare the growth after 10 hours and after $n$ hours. Comment on the difference in growth of a geometric sequence versus an arithmetic sequence.

**Other Sequences**

**Figurate numbers** provide examples of sequences that are neither arithmetic nor geometric. Such numbers can be represented by dots arranged in the shape of certain geometric figures. The number 1 is the beginning of most patterns involving figurate numbers. The array in Figure 1-15 represents the first four terms of the sequence of **triangular numbers**.

![Figure 1-15](image)

The triangular numbers can be written numerically as 1, 3, 6, 10, 15, ... The sequence 1, 3, 6, 10, 15, ... is not an arithmetic sequence because there is no common difference, as Figure 1-16 shows. It is not a geometric sequence because there is no common ratio. It is not a Fibonacci sequence.

![Figure 1-16](image)
However, the sequence of differences, 2, 3, 4, 5, . . . , is an arithmetic sequence with difference 1, as Figure 1-17 shows. The next successive terms for the original sequence are shown in color in Figure 1-17.

![Figure 1-17](image)

Table 1-12 suggests a pattern for finding the next terms and the \( n \)th term for the triangular numbers. The second term is obtained from the first term by adding 2; the third term is obtained from the second term by adding 3; and so on.

<table>
<thead>
<tr>
<th>Number of Term</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3 = 1 + 2</td>
</tr>
<tr>
<td>3</td>
<td>6 = 1 + 2 + 3</td>
</tr>
<tr>
<td>4</td>
<td>10 = 1 + 2 + 3 + 4</td>
</tr>
<tr>
<td>5</td>
<td>15 = 1 + 2 + 3 + 4 + 5</td>
</tr>
<tr>
<td></td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>.</td>
</tr>
<tr>
<td>10</td>
<td>55 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10</td>
</tr>
</tbody>
</table>

In general, because the \( n \)th triangular number has \( n \) dots in the \( n \)th row, it is equal to the sum of the dots in the previous triangular number (the \((n - 1)\)st one) plus the \( n \) dots in the \( n \)th row. Following this pattern, the 10th term is \( 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \), or 55, and the \( n \)th term, \( a_n \), is \( 1 + 2 + 3 + 4 + 5 + \ldots + (n - 1) + n \). This problem is similar to Gauss's Problem in Section 1-1. Because of the work done in Section 1-1, we know that

\[
a_n = \frac{n(n + 1)}{2}
\]

Next consider the first four square numbers in Figure 1-18. These square numbers, 1, 4, 9, 16, . . . , can be written as \( 1^2, 2^2, 3^2, 4^2 \), and so on. The number of dots in the 10th array is \( 10^2 \), the number of dots in the 100th array is \( 100^2 \), and the number of dots in the \( n \)th array is \( n^2 \). The sequence of square numbers is neither arithmetic nor geometric. Investigate whether the sequence of first differences is an arithmetic sequence and tell why.
Example 1-6

Use differences to find a pattern. Then assuming that the pattern discovered continues, find the seventh term in each of the following sequences:

a. 5, 6, 14, 29, 51, 80, ...

b. 2, 3, 9, 23, 48, 87, ...

Solution  a. Following is the sequence of first differences:

\[
\begin{array}{cccccc}
5 & 6 & 14 & 29 & 51 & 80 \\
1 & 8 & 15 & 22 & 29 \
\end{array}
\]

(First difference)

To discover a pattern for the original sequence, we try to find a pattern for the sequence of differences 1, 8, 15, 22, 29, ... . This sequence is an arithmetic sequence with fixed difference 7:

\[
\begin{array}{cccccc}
5 & 6 & 14 & 29 & 51 & 80 \\
1 & 8 & 15 & 22 & 29 \\
7 & 7 & 7 & 7 \
\end{array}
\]

(First difference)  (Second difference)

Thus, the sixth term in the first difference row is 29 + 7, or 36, and the seventh term in the original sequence is 80 + 36, or 116. What number follows 116?

b. Because the second difference is not a fixed number, we go on to the third difference:

\[
\begin{array}{cccccc}
2 & 3 & 9 & 23 & 48 & 87 \\
1 & 6 & 14 & 25 & 39 \\
5 & 8 & 11 & 14 \\
3 & 3 & 3 \
\end{array}
\]

(First difference)  (Second difference)  (Third difference)

The third difference is a fixed number; therefore, the second difference is an arithmetic sequence. The fifth term in the second-difference sequence is 14 + 3, or 17; the sixth term in the first-difference sequence is 39 + 17, or 56; and the seventh term in the original sequence is 87 + 56, or 143.
When asked to find a pattern for a given sequence, you first look for some easily recognizable pattern and determine whether the sequence is arithmetic or geometric. If a pattern is unclear, taking successive differences may help. It is possible that none of the methods described reveal a pattern.

Assessment 1-2 A

1. For each of the following sequences of figures, determine a possible pattern and draw the next figure according to that pattern:
   a.  
   b.  
   c.  

2. In each of the following, list terms that continue a possible pattern. Which of the sequences are arithmetic, which are geometric, and which are neither?
   a. 1, 3, 5, 7, 9
   b. 0, 50, 100, 150, 200
   c. 3, 6, 12, 24, 48
   d. 10, 100, 1,000, 10,000, 100,000
   e. 9, 13, 17, 21, 25, 29
   f. 1, 8, 27, 64, 125

3. Find the 100th term and the nth term for each of the sequences in exercise 2.

4. Use a traditional clock face to determine the next three terms in the following sequence:

5. In the pattern, 8, 16, 14, 10, ..., the sum of digits can be used to create the next number. In this case, each succeeding number is double the sum of the digits in the previous number.
   a. Find the next three numbers in the sequence described.
   b. Find the next three numbers in the sequence 4, 16, 49, 169, 256, ..., ____, ____, ____. Describe the rule you used.
   c. Find the next three numbers in the sequence 4, 16, 37, 58, 89, 145, 42, 20, ____, ____, ____. Describe the rule you used.
   d. What will happen if the sequence in part (c) is continued indefinitely?
6. The following geometric arrays suggest a sequence of numbers:

![Geometric Arrays](image)

a. Find the next three terms.

b. Find the 100th term.

c. Find the \( n \)th term.

7. The first windmill takes 5 matchstick squares to build, the second takes 9 to build, and the third takes 13 to build, as shown. How many matchstick squares will it take to build

(a) the 10th windmill?

(b) the \( n \)th windmill?

(c) How many matchsticks will it take to build the \( n \)th windmill?

![Windmills](image)

8. In the following sequence, the figures are made of cubes that are glued together. If the exposed surface needs to be painted, how many squares will be painted in

(a) the 10th figure?

(b) the \( n \)th figure?

![Cube Figures](image)

9. The school population for a certain school is predicted to increase by 50 students per year for the next 10 years. If the current enrollment is 700 students, what will the enrollment be after 10 years?

10. Joe’s annual income has been increasing each year by the same amount. The first year his income was $24,000, and the ninth year his income was $31,680. In which year was his income $45,120?

11. The first difference of a sequence is 2, 4, 6, 8, 10, \ldots. Find the first six terms of the original sequence in each of the following cases:

(a) The first term of the original sequence is 3.

(b) The sum of the first two terms of the original sequence is 10.

(c) The fifth term of the original sequence is 35.

12. List the next three terms to continue a pattern in each of the following. (Finding differences may be helpful.)

(a) 5, 6, 14, 32, 64, 115, 191

(b) 0, 2, 6, 12, 20, 30, 42

13. How many terms are there in each of the following sequences?

(a) 51, 52, 53, 54, \ldots, 151

(b) 1, 2, 2^2, 2^3, \ldots, 2^{60}

(c) 10, 20, 30, 40, \ldots, 2000

(d) 1, 2, 4, 8, 16, 32, \ldots, 1024

14. Find the first five terms in each of the following:

(a) \( a_n = n^2 + 2 \)

(b) \( a_n = 5n - 1 \)

(c) \( a_n = 10^n - 1 \)

15. Find a counterexample for each of the following:

(a) If \( x \) is a natural number, then \((x + 5)/5 = x + 1\).

(b) If \( x \) is a natural number, then \((x + 4)^2 = x + 16\).

16. Assume that the following pattern of square tile figures, (□), continues and answer the questions that follow.

![Square Tile Figures](image)

(a) How many square tiles are there in the sixth figure, \( a_6 \)?

(b) How many square tiles are in the \( n \)th shape, \( a_n \)?

(c) Is there a figure that has exactly 1259 square tiles? If so, which one?

17. Find the third, fourth, and fifth terms in the sequence if

\( a_1 = 2, a_2 = 5, \) and \( a_n = 2a_{n-1} - a_{n-2} \).

18. Consider the following sequences:

\[
\begin{align*}
300, 500, 700, 900, 1100, 1300, \ldots \\
2, 4, 8, 16, 32, 64, \ldots
\end{align*}
\]

Find the number of the term in which the geometric sequence becomes greater than the arithmetic sequence.

19. Start out with a piece of paper. Next, cut the piece of paper into five pieces. Take any one of the pieces and cut it into five pieces, and so on.

(a) What number of pieces can be obtained in this way?

(b) What is the total number of pieces obtained after \( n \)th cuts?

20. The sequence 32, \( a, b, c, 512, \ldots \) is a geometric sequence. Find \( a, b, c \).

21. Assume the following pattern of dots continues:

(a) How many dots are there in \( a_6 \)?

(b) How many dots are in the \( n \)th shape, \( a_n \)?

![Dots Pattern](image)
Assessment 1-2 B

1. For each of the following sequence of figures, determine a possible pattern and draw the next figure according to that pattern:
   a. [Diagram]
   b. [Diagram]
   c. [Diagram]

2. In each of the following, list terms that continue a possible pattern. Which of the sequences are arithmetic, which are geometric, and which are neither?
   a. 8, 11, 14, 17, 20, ...
   b. 1, 16, 81, 256, 625, ...
   c. 5, 15, 45, 135, 405, ...
   d. 2, 7, 12, 17, 22, ...
   e. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

3. Find the 100th term and the $n$th term for each of the sequences in exercise 2.

4. Observe the following pattern:
   
   \[
   1 + 3 = 2^2, \\
   1 + 3 + 5 = 3^2, \\
   1 + 3 + 5 + 7 = 4^2
   \]
   a. State a generalization based on this pattern.
   b. Based on the generalization in (a), find $1 + 3 + 5 + 7 + \ldots + 35$

5. In the following pattern, one hexagon takes 6 toothpicks to build, two hexagons take 11 toothpicks to build, and so on. How many toothpicks would it take to build (a) 10 hexagons? (b) $n$ hexagons?

6. Each of the following figures is made of small triangles like the first one in the sequence. (The second figure is made of four small triangles.) Make a conjecture concerning the number of small triangles needed to make (a) the 100th figure and (b) the $n$th figure.

7. At the end of the day, a tank contains 15,360 L of water. At the end of each subsequent day, half of the water is removed and not replaced. How much water is left in the tank after 10 days?

8. Each side of each pentagon below is 1 unit long.
   a. Draw a next figure in the sequence.
   b. What is the perimeter (distance around) of each of the first four figures?
   c. What is the perimeter of the 100th figure?
   d. What is the perimeter of the $n$th figure?

9. Washington Middle School has a schedule that forms an arithmetic sequence. Each period is the same length and includes a 4th period lunch. The first three periods begin at 8:10 A.M., 9:00 A.M., and 9:50 A.M., respectively. At what time does the eighth period begin?

10. The first difference of a sequence is 3, 6, 9, 12, 15, ... Find the first six terms of the original sequence in each of the following cases:
   a. The first term of the original sequence is 3.
   b. The sum of the first two terms of the original sequence is 7.
   c. The fifth term of the original sequence is 34.

11. List the next three terms to continue a pattern in each of the following. (Finding differences may be helpful.)
   a. 3, 8, 15, 24, 35, 48, ...
   b. 1, 7, 18, 37, 67, 111, ...

12. How many terms are there in each of the following sequences?
   a. 1, 2, 2^2, 2^3, \ldots, 2^{60}
   b. 9, 13, 17, 21, 25, \ldots, 353
   c. 38, 39, 40, 41, \ldots, 198

13. Find the first five terms in each of the following:
   a. $a_n = 5n + 6$
   b. $a_n = 6n - 2$
   c. $a_n = 5^n + 1$
   d. $a_n = 3n - 3$

14. Find a counterexample for each of the following:
   a. If $x$ is a natural number, then $(3 + x)/3 = x$.
   b. If $x$ is a natural number, then $(x - 2)^2 = x^2 - 2^2$.

15. Assume the following pattern of square tile figures, (☐), continues and answer the questions that follow.
An Introduction to Problem Solving

1. How many square tiles are there in the sixth figure, \( a_6 \)?
2. How many square tiles are in the \( n \)th shape, \( a_n \)?
3. Is there a figure that has exactly 449 square tiles? If so, which one?

16. Write consecutive odd numbers in triangular form as shown.

\[
\begin{array}{ccc}
1 \\
3 & 5 \\
7 & 9 & 11 \\
13 & 15 & 17 & 19 \\
& & & & \\
\end{array}
\]

and so on

a. Find the sum of each of the first five horizontal rows.
b. What patterns do you notice?

17. Find the third, fourth, and fifth terms in the sequence if 
\[ a_1 = 3, \quad a_2 = 6, \quad \text{and} \quad a_n = 3a_{n-1} - 2a_{n-2}. \]

18. Consider the following sequences:

\[
\begin{align*}
200, & \quad 500, & \quad 800, & \quad 1100, & \quad 1400, & \quad 1700, & \ldots \\
1, & \quad 3, & \quad 9, & \quad 27, & \quad 81, & \quad 243, & \ldots 
\end{align*}
\]

Find the number of the term after which the geometric sequence becomes greater than the arithmetic sequence.

19. The sequence \( 17, \quad a, \quad b, \quad c, \quad 1377, \ldots \) is a geometric sequence. Find \( a, b, c \).

20. Find the sum of the first 43 terms of an arithmetic sequence in which the 11th term is 83 and the 62nd term is 440.

21. Female bees are born from fertilized eggs, and male bees are born from unfertilized eggs. This means that a male bee has only a mother, whereas a female bee has a mother and a father. If the ancestry of a male bee is traced 10 generations, how many bees are there in all 10 generations? (Hint: The Fibonacci sequence might be helpful.) Explain how you arrived at your answer.

Mathematical Connections 1-2

Communication

1. Explain how the two sequences in each part are the same and how they are different.
   a. 2, 4, 6, 8, 10, \ldots \quad and \quad 2, 4, 8, 16, 32, \ldots
   b. 2, 4, 6, 8, 10, \ldots \quad and \quad 3, 5, 7, 9, 11, \ldots
   c. 5, 10, 15, 20, 25, \ldots \quad and \quad 50, 100, 150, 200, 250, \ldots

2. Give two examples of how inductive reasoning might be used in everyday life. Is a conclusion based on inductive reasoning certain?

3. a. If a fixed number is added to each term of an arithmetic sequence, is the resulting sequence an arithmetic sequence? Justify the answer.
   b. If each term of an arithmetic sequence is multiplied by a fixed number, will the resulting sequence always be an arithmetic sequence? Justify the answer.
   c. If the corresponding terms of two arithmetic sequences are added, is the resulting sequence arithmetic?

4. A student says she read that Thomas Robert Malthus (1766–1834), a renowned British economist and demographer, claimed that the increase of population will take place, if unchecked, in a geometric sequence, whereas the supply of food will increase in only an arithmetic sequence. This theory implies that population increases faster than food production. The student is wondering why. How do you respond?

Open-Ended

5. Patterns can be used to count the number of dots on the Chinese checkerboard; two patterns are shown here. Determine several other patterns to count the dots.

\[
\begin{align*}
1 + 2 + 3 + \ldots + 13 + 3(10) \\
1 + 3 + 5 + 7 + \ldots + 17 + 4(10)
\end{align*}
\]

6. Make up a pattern involving figurate numbers and find a formula for the 100th term. Describe the pattern and how to find the 100th term.

7. A sequence that follows the same pattern as the Fibonacci sequence but in which the first two terms are not 1s but any numbers is called a Fibonacci type sequence. Choose a few such sequences and answer the questions in Now Try This 1-13. Do these sequences behave in the same way?
Cooperative Learning

8. The following pattern is called Pascal's triangle. It was named for the mathematician Blaise Pascal (1623–1662).

```
  1
 1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
```

a. Have each person in the group find four different patterns in the triangle and then share them with the rest of the group.
b. Add the numbers in each horizontal row. Discuss the pattern that occurs.
c. Use what you have learned in part (b) to find the sum in the 16th row.
d. What is the sum of the numbers in the $n$th row?

9. If the following pattern continued indefinitely, the resulting figure would be called the Sierpinski triangle, or Sierpinski gasket.

In a group, determine each of the following. Discuss different counting strategies.
a. How many black triangles would be in the fifth figure?
b. How many white triangles would be in the fifth figure?
c. If the pattern is continued for $n$ figures, how many black triangles will there be?
d. If the pattern is continued for $n$ figures, how many white triangles will there be?

10. Create your own sequence of numbers that follow a pattern. Share your sequence with the other members of your group. If they can't determine a correct rule for your pattern, explain it to them.

Questions from the Classroom

11. Joey said that 4, 24, 44, and 64 all have remainder 0 when divided by 4, so all numbers that end in 4 must have 0 remainder when divided by 4. How do you respond?

12. Al and Betty were asked to extend the sequence 2, 4, 8, . . . . Al said his answer of 2, 4, 8, 16, 32, 64, . . . was the correct one. Betty said Al was wrong and it should be 2, 4, 8, 14, 22, 32, 44, . . . What do you tell these students?

13. A student claims that if both the numerator and denominator of a fraction are greater than the numerator and denominator, respectively, of another fraction, then the first fraction must be the greater of the two. How do you respond?

14. A student claims the sequence 6, 6, 6, 6, 6, . . . never changes, so it is neither arithmetic nor geometric. How do you respond?

15. A student claims that two terms are enough to determine any sequence. For example, 3, 6, . . . means the sequence is 3, 6, 9, 12, 15, . . . How do you respond?

16. Lisa claims subtracting the first term from the last term and dividing by the common difference can find the number of terms in any finite arithmetic sequence. How do you respond?

Review Problems

17. In a baseball league consisting of 10 teams, each team plays each of the other teams twice. How many games will be played?

18. How many ways can you make change for using only nickels, dimes, and quarters?

19. Tents hold 2, 3, 5, 6, or 12 people. What combinations of tents are possible to sleep 26 people if only one 12-person tent is used?

Third International Mathematics and Science Study (TIMSS) Questions

The numbers in the sequence 7, 11, 15, 19, 23, . . . increase by four. The numbers in the sequence 1, 10, 19, 28, 37, . . . increase by nine. The number 19 is in both sequences. If the two sequences are continued, what is the next number that is in BOTH the first and the second sequences?

**TIMSS 2003, Grade 8**

The three figures below are divided into small congruent triangles.

```
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
</tr>
</thead>
</table>
```

```
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
</table>
```

```
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
</table>
```

Figure 1  Figure 2  Figure 3
a. Complete the table below. First, fill in how many small triangles make up Figure 3. Then, find the number of small triangles that would be needed for the fourth figure if the sequence of figures is extended.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Numbers of Small Triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

b. The sequence of figures is extended to the 7th figure. How many small triangles would be needed for Figure 7?

c. The sequence of figures is extended to the 50th figure. Explain a way to find the number of small triangles in the 50th figure that does not involve drawing it and counting the number of triangles.

TIMSS, Grade 8

---

**BRAIN TEASER** Find the next row in the following pattern and explain your pattern:

```
1
1 1
2 1
1 2 1 1
1 1 2 2 1
```

---

**Reasoning and Logic: An Introduction**

Logic is a tool used in mathematical thinking and problem solving. It is essential for reasoning and, as pointed out in the Research Note, cannot be taught in a single unit on logic. However, we present a very brief view of some “basics” in this section. In logic, a **statement** is a sentence that is either true or false, but not both. The following expressions are not statements because their truth values cannot be determined without more information:

1. She has blue eyes.
2. $x + 7 = 18$.
3. $2y + 7 > 1$.
4. $2 + 3$

Expressions (1), (2), and (3) become statements if, for (1), “she” is identified, and for (2) and (3), values are assigned to $x$ and $y$, respectively. However, an expression involving he or she or $x$ or $y$ may already be a statement. For example, “If he is over 210 cm tall, then he is over 2 m tall” and “$2(x + y) = 2x + 2y$” are both statements because they are true no matter who he is or what the numerical values of $x$ and $y$ are.

**Negation and Quantifiers**

From a given statement, it is possible to create a new statement by forming a **negation**. The negation of a statement is a statement with the opposite truth value of the given statement. If a statement is true, its negation is false, and if a statement is false, its negation is true. Consider the statement “It is snowing now.” The negation of this statement may be stated simply as “It is not snowing now.”

---

*Research Note*

Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by “doing proofs” in geometry. Proof is a very difficult area for undergraduate mathematics students. Perhaps students at the post-secondary level find proof so difficult because their only experience in writing proofs has been in a high school geometry course, so they have a limited perspective (Moore 1994).
Example 1-7
Negate each of the following statements:

a. $2 + 3 = 5$.

b. A hexagon has six sides.

**Solution**

a. $2 + 3 \neq 5$.

b. A hexagon does not have six sides.

Example 1-8

Negate each of the following regardless of its truth value:

a. All students like hamburgers.

b. Some people like mathematics.

c. There exists a natural number $x$ such that $3x = 6$.

d. For all natural numbers, $3x = 3x$. 

---

Sentences like “The shirt is blue” and “The shirt is green” are statements if put in context. However, they are not negations of each other. A statement and its negation must have opposite truth values in all possible cases. If the shirt is actually red, then both of the statements are false and, hence, cannot be negations of each other. However, the statements “The shirt is blue” and “The shirt is not blue” are negations of each other because they have opposite truth values no matter what color the shirt really is.

Some statements involve quantifiers and are more complicated to negate. Quantifiers include words such as *all*, *some*, *every*, and *there exists*.

- The quantifiers *all*, *every*, and *no* refer to each and every element in a set and are called *universal quantifiers*.
- The quantifiers *some* and *there exists at least one* refer to one or more, or possibly all, of the elements in a set and are called *existential quantifiers*.
- *All*, *every*, and *each* have the same mathematical meaning. Similarly, *some* and *there exists at least one* both have the same meaning.

Consider the following statement involving the existential quantifier *some* and known to be true: “Some professors at Paxson University have blue eyes.” This means that at least one professor at Paxson University has blue eyes. It does not rule out the possibilities that *all* the Paxson professors have blue eyes or that some of the Paxson professors do not have blue eyes. Because the negation of a true statement is false, neither “Some professors at Paxson University do not have blue eyes” nor “All professors at Paxson have blue eyes” are negations of the original statement. One possible negation of the original statement is “No professors at Paxson University have blue eyes.”

To discover if one statement is a negation of another, we use arguments similar to the preceding one to determine if they have opposite truth values in all possible cases.

General forms of quantified statements with their negations follow:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Some $a$ are $b$.</td>
<td>No $a$ is $b$.</td>
</tr>
<tr>
<td>Some $a$ are not $b$.</td>
<td>All $a$ are $b$.</td>
</tr>
<tr>
<td>All $a$ are $b$.</td>
<td>Some $a$ are not $b$.</td>
</tr>
<tr>
<td>No $a$ is $b$.</td>
<td>Some $a$ are $b$.</td>
</tr>
</tbody>
</table>

---

An Introduction to Problem Solving

Solution

a. Some students do not like hamburgers.
b. No people like mathematics.
c. For all natural numbers \( x \), \( 3x \neq 6 \).
d. There exists a natural number \( x \) such that \( 3x \neq 3x \).

Truth Tables and Compound Statements

To investigate the truth of statements, consider the following puzzle by one of today’s foremost writers of logic puzzles, Raymond Smullyan. He has written several books on logic, including The Lady or the Tiger? This title is taken from the Frank Stockton short story about a prisoner who must make a choice between two doors: Behind one is a hungry tiger and behind the other is a beautiful lady.

Smullyan proposes that each door has a sign and the prisoner knows that only one sign is true. The sign on Door 1 reads:

IN THIS ROOM THERE IS A LADY AND
IN THE OTHER ROOM THERE IS A TIGER.

The sign on Door 2 reads:

IN ONE OF THESE ROOMS THERE IS A LADY AND
IN ONE OF THESE ROOMS THERE IS A TIGER.

With this information, the man can choose the correct door. Discuss this problem and try to find a solution before reading on.

Solution

If the sign on Door 1 is true, then the sign on Door 2 must be true. Since this cannot happen, the sign on Door 2 must be true, making the sign on Door 1 false. Because the sign on Door 1 is false, the lady can’t be in Room 1 and must be in Room 2.

There is a symbolic system defined to help in the study of logic. If \( p \) represents a statement, the negation of the statement \( p \) is denoted by \( \sim p \) and is read “not \( p \).” Truth tables are often used to show all possible true-false patterns for statements. Table 1-13 summarizes the truth tables for \( p \) and \( \sim p \).

From two given statements, it is possible to create a new, compound statement by using a connective such as \( \text{and} \). A compound statement may be formed by combining two or more statements. For example, “It is snowing” and “The ski run is open” together with \( \text{and} \) give “It is snowing and the ski run is open.” Other compound statements can be obtained by using the connective \( \text{or} \). For example, “It is snowing or the ski run is open.” The symbols \( \wedge \) and \( \vee \) are used to represent the connectives \( \text{and} \) and \( \text{or} \), respectively. For example, if \( p \) represents “It is snowing” and \( q \) represents “The ski run is open,” then “It is snowing and the ski run is open” is denoted by \( p \wedge q \). Similarly, “It is snowing or the ski run is open” is denoted by \( p \vee q \).

The truth value of any compound statement, such as \( p \wedge q \), is defined using the truth value of each of the simple statements. Because each of the statements \( p \) and \( q \) may be either true or false, there are four distinct possibilities for the truth value of \( p \wedge q \), as shown

<table>
<thead>
<tr>
<th>Statement</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( \sim p )</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
in Table 1-14. The compound statement is the **conjunction** of \( p \) and \( q \) and is defined to be true if, and only if, both \( p \) and \( q \) are true. Otherwise, it is false.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>Conjunction ( p \land q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

The compound statement \( p \lor q \)—that is, \( p \text{ or } q \)—is a **disjunction**. In everyday language, \( \text{or} \) is not always interpreted in the same way. In logic, we use an **inclusive or**. The statement “I will go to a movie or I will read a book” means I will either go to a movie, or read a book, or do both. Hence, in logic, \( p \text{ or } q \), symbolized \( p \lor q \), is defined to be false if both \( p \) and \( q \) are false and true in all other cases. This is summarized in Table 1-15.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>Disjunction ( p \lor q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**Example 1-9**

Classify each of the following as true or false:

\[
p: 2 + 3 = 5 \quad q: 2 \cdot 3 = 6 \quad r: 5 + 3 = 9
\]

a. \( p \land q \)  
   b. \( q \lor r \)  
   c. \( \neg p \lor r \)  
   d. \( \neg p \land \neg q \)  
   e. \( \neg(p \land q) \)  
   f. \( (p \land q) \lor \neg r \)

**Solution**

a. \( p \) is true and \( q \) is true, so \( p \land q \) is true.
   
b. \( q \) is true and \( r \) is false, so \( q \lor r \) is true.
   
c. \( \neg p \) is false and \( r \) is false, so \( \neg p \lor r \) is false.
   
d. \( \neg p \) is false and \( \neg q \) is false, so \( \neg p \land \neg q \) is false.
   
e. \( p \land q \) is true so \( \neg(p \land q) \) is false.
   
f. \( p \land q \) is true and \( \neg r \) is true, so \( (p \land q) \lor \neg r \) is true.

Truth tables are used not only to summarize the truth values of compound statements; they also are used to determine if two statements are logically equivalent. Two statements are **logically equivalent** if, and only if, they have the same truth values in every possible situation. If \( p \) and \( q \) are logically equivalent, we write \( p \equiv q \).

---

**Historical Note**

George Boole (1815–1864), born in Lincoln, England, is called “the father of logic.” At age 15, he began a teaching career. In 1849, he was appointed professor at Queens College in Cork, Ireland. In his work he employed symbols to represent concepts and developed a system of algebraic manipulations to accompany the symbols. His work was a marriage of logic and mathematics. Many of Boole’s ideas, such as Boolean algebra, have applications in computer science and in the design of telephone switching devices.
Example 1-10  
Show that \( \sim(p \land q) \equiv \sim p \lor \sim q \).

Solution  
Two statements are logically equivalent if they have the same truth values. Truth tables for these statements are given in Table 1-16 and Table 1-17.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \land q )</th>
<th>( \sim(p \land q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \sim p )</th>
<th>( \sim q )</th>
<th>( \sim p \lor \sim q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Because the two statements have the same truth values, we know that \( \sim(p \land q) \equiv \sim p \lor \sim q \).

NOW TRY THIS 1-16  
Example 1-10 shows that \( \sim(p \land q) = \sim p \lor \sim q \). In the same way we can show that \( \sim(p \lor q) = \sim p \land \sim q \). We call these equivalencies De Morgan’s Laws. Use truth tables to confirm the second De Morgan Law.

Conditionals and Biconditionals

Statements expressed in the form “if \( p \), then \( q \)” are conditionals, or implications, and are denoted by \( p \rightarrow q \). Such statements also can be read “\( p \) implies \( q \)” The “if” part of a conditional is the hypothesis of the implication and the “then” part is the conclusion. Many types of statements can be put in “if-then” form. An example follows:

Statement: All equilateral triangles have acute angles.
If-then form: If a triangle is equilateral, then it has acute angles.

Hypothesis  Conclusion

An implication may also be thought of as a promise. Suppose Betty makes the promise “If I get a raise, then I will take you to dinner.” If Betty keeps her promise, the implication is true; if Betty breaks her promise, the implication is false. Consider the following four possibilities:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) T</td>
<td>T</td>
<td>Betty gets the raise; she takes you to dinner.</td>
</tr>
<tr>
<td>(2) T</td>
<td>F</td>
<td>Betty gets the raise; she does not take you to dinner.</td>
</tr>
<tr>
<td>(3) F</td>
<td>T</td>
<td>Betty does not get the raise; she takes you to dinner.</td>
</tr>
<tr>
<td>(4) F</td>
<td>F</td>
<td>Betty does not get the raise; she does not take you to dinner.</td>
</tr>
</tbody>
</table>

The only case in which Betty breaks her promise is when she gets her raise and fails to take you to dinner, case (2). If she does not get the raise, she can either take you to dinner or not without breaking her promise. The definition of implication is summarized in
Table 1-18. Observe that the only case for which the implication is false is when \( p \) is true and \( q \) is false.

An implication can be worded in several equivalent ways, as follows:

1. If the sun shines, then the swimming pool is open. (If \( p \), then \( q \).)
2. If the sun shines, the swimming pool is open. (If \( p \), \( q \).)
3. The swimming pool is open if the sun shines. (\( q \) if \( p \).)
4. The sun is shining implies the swimming pool is open. (\( p \) implies \( q \)).
5. The sun is shining only if the pool is open. (\( p \) only if \( q \)).
6. The sun's shining is a sufficient condition for the swimming pool to be open. (\( p \) is a sufficient condition for \( q \)).
7. The swimming pool's being open is a necessary condition for the sun to be shining. (\( q \) is a necessary condition for \( p \)).

Any implication \( p \rightarrow q \) has three related implication statements, as follows:

<table>
<thead>
<tr>
<th>Statement: If ( p ), then ( q ).</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converse: If ( q ), then ( p ).</td>
<td>( q \rightarrow p )</td>
</tr>
<tr>
<td>Inverse: If not ( p ), then not ( q ).</td>
<td>( \sim p \rightarrow \sim q )</td>
</tr>
<tr>
<td>Contrapositive: If not ( q ), then not ( p ).</td>
<td>( \sim q \rightarrow \sim p )</td>
</tr>
</tbody>
</table>

Example 1-11 can be used to show that if an implication is true, its converse and inverse are not necessarily true. However, the contrapositive is true. Let's check these observations on the following true statement: If a number is a natural number, the number is not 0. The set of natural numbers are the numbers \( N = \{1, 2, 3, 4, 5, 6, \ldots \} \). We check the truth of the converse, inverse, and contrapositive.

Inverse: If a number is not a natural number, then it is 0. This is false, since \(-6\) is not a natural number but it also is not 0.
Converse: If a number is not 0, then it is a natural number. This is false, since \(-6\) is not 0 but neither is it a natural number.
Contrapositive: If a number is 0, then it is not a natural number. This is true because \( N = \{1, 2, 3, 4, 5, 6, \ldots \} \).

The contrapositive of the last statement is the original statement. Hence, the preceding discussion suggests that if \( p \rightarrow q \) is true, its contrapositive \( \sim q \rightarrow \sim p \) is also true, and if the contrapositive is true, the original statement must be true. It follows that a statement and its contrapositive cannot have opposite truth values. We summarize this in the following theorem.
Connecting a statement and its converse with the connective \( \text{and} \) gives \( (p \rightarrow q) \land (q \rightarrow p) \). This compound statement can be written as \( p \leftrightarrow q \) and usually is read “\( p \text{ if, and only if, } q \).” The statement “\( p \text{ if, and only if, } q \)” is a biconditional.

Valid Reasoning

In problem solving, the reasoning is said to be valid if the conclusion follows unavoidably from true hypotheses. Thus, in all arguments in this section we assume the hypotheses are true. Consider the following examples:

Hypotheses: All dogs are animals.
Goofy is a dog.

Conclusion: Therefore, Goofy is an animal.
The statement “All dogs are animals” can be pictured with the Euler diagram in Figure 1-20(a).

![Figure 1-20](a)

The information “Goofy is a dog” implies that Goofy now also belongs to the circle containing dogs, as pictured in Figure 1-20(b). Goofy must also belong to the circle containing animals. Thus, the reasoning is valid because it is impossible to draw a picture that satisfies the hypotheses and contradicts the conclusion.

Consider the following argument.

Hypotheses: All elementary schoolteachers are mathematically literate.
Some mathematically literate people are not children.

Conclusion: Therefore, no elementary schoolteacher is a child.

Let $E$ be the set of elementary schoolteachers, $M$ be the set of mathematically literate people, and $C$ be the set of children. Then the statement “All elementary schoolteachers are mathematically literate” can be pictured as in Figure 1-21(a). The statement “Some mathematically literate people are not children” can be pictured in several ways. Three of these are illustrated in Figure 1-21(b) through (d). According to Figure 1-21(d), it is possible that some elementary schoolteachers are children, and yet the given statements are satisfied. Therefore, the conclusion that “No elementary schoolteacher is a child” does not follow from the given hypotheses. Hence, the reasoning is not valid.

![Figure 1-21](a)

If even one picture can be drawn to satisfy the hypotheses of an argument and contradict the conclusion, the argument is not valid. However, to show that an argument is valid, all possible pictures must show that there are no contradictions. There must be no way to satisfy the hypotheses and contradict the conclusion if the argument is valid.
Determine whether the following argument is valid if the hypotheses are true and $x$ is a natural number:

**Hypotheses:**

- In Washington, D.C., all lobbyists wear suits.
- No one in Washington, D.C., over 6 ft tall wears a suit.

**Conclusion:**

- Persons over 6 ft tall are not lobbyists in Washington, D.C.

**Solution**

Using the law of detachment, modus ponens, we see that the conclusion is valid.

$x^2 > 7^2$

$x > 2$.

---

**Example 1-14**

Determine whether the following argument is valid if the hypotheses are true and $x$ is a natural number:

**Hypotheses:**

- If $x > 2$, then $x^2 > 4$.
- $x > 2$.

**Conclusion:**

- Therefore, $x^2 > 4$.

**Solution**

Using the law of detachment, modus ponens, we see that the conclusion is valid.

---

**Example 1-15**

Show that $[(p \rightarrow q) \land p] \rightarrow q$ is always true.
**Solution** A truth table for this implication is given in Table 1-21.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
<th>$(p \rightarrow q) \land p$</th>
<th>$[(p \rightarrow q) \land p] \rightarrow q$</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

**Remark** The statement $[(p \rightarrow q) \land p] \rightarrow q$ is a tautology; that is, a statement that is true all the time.

A different type of reasoning, indirect reasoning, uses a form of argument called *modus tollens*. For example, consider the following true statements:

If a figure is a square, then it is a rectangle.
The figure is not a rectangle.

The conclusion is that the figure cannot be a square. *Modus tollens* can be interpreted as follows:

If we have a conditional accepted as true, and we know the conclusion is false, then the hypothesis must be false.

**Example 1-16** Determine conclusions for each of the following pairs of true statements:

a. If a person lives in Boston, then the person lives in Massachusetts. Jessica does not live in Massachusetts.

b. If $x = 3$, then $2x \neq 7$. We know that $2x = 7$.

**Solution**

a. Jessica does not live in Boston (*modus tollens*).

b. $x \neq 3$ (*modus tollens*).

The final reasoning argument we consider here involves the chain rule (transitivity). Consider the following statements:

If I save, I will retire early.
If I retire early, I will play golf.

What is the conclusion? The conclusion is that if I save, I will play golf. In general, the chain rule can be stated as follows:

If “if $p$, then $q$” and “if $q$, then $r$” are true, then “if $p$, then $r$” is true.

People often make invalid conclusions based on advertising or other information. Assume, for example, the statement “Healthy people eat Super-Bran cereal” is true. Are the following conclusions valid?
If a person eats Super-Bran cereal, then the person is healthy.
If a person is not healthy, the person does not eat Super-Bran cereal.

If the original statement is denoted by \( p \rightarrow q \), where \( p \) is “a person is healthy” and \( q \) is “a person eats Super-Bran cereal,” then the first conclusion is the converse of \( p \rightarrow q \), that is, \( q \rightarrow p \), and the second conclusion is the inverse of \( p \rightarrow q \), that is, \( \sim p \rightarrow \sim q \). Hence, neither is valid.

Example 1-17

Determine valid conclusions for the following true statements:

a. If a triangle is equilateral, then it is isosceles. If a triangle is isosceles, it has at least two congruent sides.
b. If a number is a whole number, then the number is an integer. If a number is an integer, then the number is a rational number. If a number is a rational number, then the number is a real number.

Solution  

a. If a triangle is equilateral, then it has at least two congruent sides.
b. If a number is a whole number, then it is a real number.

Assessment 1-3A

1. Determine which of the following are statements and then classify each statement as true or false:
   a. \( 2 + 4 = 8 \).
   b. Los Angeles is a state.
   c. What time is it?
   d. \( 3 \cdot 2 = 6 \).
   e. This statement is false.

2. Use quantifiers to make each of the following true, where \( x \) is a natural number:
   a. \( x + 8 = 11 \).
   b. \( x^2 = 4 \).
   c. \( x + 3 = 3 + x \).
   d. \( 5x + 4x = 9x \).

3. Use quantifiers to make each equation in problem 2 false.

4. Write the negation of each of the following statements:
   a. This book has 500 pages.
   b. \( 3 \cdot 5 = 15 \).
   c. All dogs have four legs.
   d. Some rectangles are squares.
   e. Not all rectangles are squares.
   f. No dogs have fleas.

5. Identify the following as true or false:
   a. For some natural numbers \( x, x < 6 \) and \( x > 3 \).
   b. For all natural numbers \( x, x > 0 \) or \( x < 5 \).

6. Complete each of the following truth tables:
   \[
   \begin{array}{c|c|c}
   p & \sim p & \sim(\sim p) \\
   \hline
   T & F & T \\
   \end{array}
   \]

   \[
   \begin{array}{c|c|c|c|c}
   p & \sim p & p \lor \sim p & p \land \sim p \\
   \hline
   T & F & T & F \\
   \end{array}
   \]

   c. Based on part (a), is \( p \) logically equivalent to \( \sim(\sim p) \)?
   d. Based on part (b), is \( p \lor \sim p \) logically equivalent to \( p \land \sim p \)?

7. If \( q \) stands for “This course is easy” and \( r \) stands for “Lazy students do not study,” write each of the following in symbolic form:
   a. This course is easy, and lazy students do not study.
   b. Lazy students do not study, or this course is not easy.
   c. It is false that both this course is easy and lazy students do not study.
   d. This course is not easy.
8. If $p$ is false and $q$ is true, find the truth values for each of the following:
   a. $p \land q$
   b. $\neg p$
   c. $\neg(p \lor q)$
   d. $p \land \neg q$
   e. $\neg(p \land q)$

9. Find the truth value for each statement in problem 8 if $p$ is false and $q$ is false.

10. For each of the following, is the pair of statements logically equivalent?
   a. $\neg(p \lor q)$ and $\neg p \land \neg q$
   b. $\neg(p \land q)$ and $\neg p \land \neg q$

11. Complete the following truth table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg p \land q$</th>
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</table>

12. Write each of the following in symbolic form if $p$ is the statement “It is raining” and $q$ is the statement “The grass is wet.”
   a. If it is raining, then the grass is wet.
   b. If it is not raining, then the grass is wet.
   c. If it is raining, then the grass is not wet.
   d. The grass is wet if it is raining.
   e. The grass is not wet implies that it is not raining.
   f. The grass is wet if, and only if, it is raining.

13. For each of the following implications, state the converse, inverse, and contrapositive:
   a. If $x = 5$, then $2x = 10$.
   b. If you do not like this book, then you do not like mathematics.
   c. If you do not use Ultra Brush toothpaste, then you have cavities.
   d. If you are good at logic, then your grades are high.

14. Consider the statement “If every digit of a number is 6, then the number is divisible by 3.” Which of the following is logically equivalent to the statement?

   a. If every digit of a number is not 6, then the number is not divisible by 3.
   b. If a number is not divisible by 3, then some digit of the number is not 6.
   c. If a number is divisible by 3, then every digit of the number is 6.

15. Write a statement logically equivalent to the statement “If a number is a multiple of 8, then it is a multiple of 4.”

16. Investigate the validity of each of the following arguments:
   a. All squares are quadrilaterals.
   All quadrilaterals are polygons.
   Therefore, all squares are polygons.
   b. All teachers are intelligent.
   Some teachers are rich.
   Therefore, some intelligent people are rich.
   c. If a student is a freshman, then the student takes mathematics.
   Jane is a sophomore.
   Therefore, Jane does not take mathematics.

17. For each of the following, form a conclusion that follows logically from the given statements:
   a. Some freshmen like mathematics.
   All people who like mathematics are intelligent.
   b. If I study for the final, then I will pass the final.
   If I pass the final, then I will pass the course.
   If I pass the course, then I will look for a teaching job.
   c. Every equilateral triangle is isosceles.
   There exist triangles that are equilateral.

18. Write the following in if-then form:
   a. Every figure that is a square is a rectangle.
   b. All integers are rational numbers.
   c. Polygons with exactly three sides are triangles.

19. Use De Morgan’s Laws from Now Try This 1-16 to write a negation of each of the following:
   a. $3 \cdot 2 = 6$ and $1 + 1 \neq 3$.
   b. You can pay me now or you can pay me later.
3. Use quantifiers to make each equation in problem 2 false.

4. Write the negation of each of the following statements:
   a. Six is less than 8.
   b. Some cats do not have nine lives.
   c. All squares are rectangles.
   d. Not all numbers are positive.
   e. Some people have blond hair.

5. Identify the following as true or false:
   a. For some natural numbers \( x, x > 5 \) and \( x > 2 \).
   b. For all natural numbers \( x, x > 5 \) or \( x < 5 \).

6. If you know that \( p \) is true, what can you conclude about the truth value of \( p \lor q \), even if you don’t know the truth value of \( q \)?

7. If you know that \( p \) is false, what can you conclude about the truth value of \( p \rightarrow q \), even if you don’t know the truth value of \( q \)?

8. If \( q \) stands for “You said goodbye” and \( r \) stands for “I said hello,” write each of the following in symbolic form:
   a. You said goodbye and I said hello.
   b. You said goodbye and I did not say hello.
   c. I did not say hello or you did not say goodbye.
   d. It is false that both you said goodbye and I said hello.

9. If \( p \) is false and \( q \) is true, find the truth values for each of the following:
   a. \( p \lor q \)
   b. \( \sim q \)
   c. \( \sim p \lor q \)
   d. \( \sim (p \lor q) \)
   e. \( \sim q \land \sim p \)

10. Find the truth value for each statement in problem 9 if \( p \) is false and \( q \) is false.

11. For each of the following, is the pair of statements logically equivalent?
    a. \( \sim (p \lor q) \) and \( \sim p \land \sim q \)
    b. \( \sim (p \land q) \) and \( \sim p \lor \sim q \)

12. Complete the following truth table:

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13. Write each of the following in symbolic form if \( p \) is the statement “You build it” and \( q \) is the statement “They will come”:
   a. If you build it, they will come.
   b. If you do not build it, then they will come.
   c. If you build it, they will not come.
   d. They will come if you build it.
   e. If you do not build it, then they will not come.
   f. If they will not come, then you do not build it.

14. For each of the following implications, state the converse, inverse, and contrapositive:
   a. If \( x = 3 \), then \( x^2 = 9 \).
   b. If it snows, then classes are canceled.

15. Iris makes the true statement “If it rains, then I am going to the movies.” Does it follow logically that if it does not rain, then Iris does not go to the movies?

16. Investigate the validity of each of the following arguments:
   a. All women are mortal.
      Hypatia was a woman.
      Therefore, Hypatia was mortal.
   b. All rainy days are cloudy.
      Today is not cloudy.
      Therefore, today is not rainy.
   c. Some students like skiing.
      Al is a student.
      Therefore, Al likes skiing.

17. For each of the following, form a conclusion that follows logically from the given statements:
   a. All college students are poor.
      Helen is a college student.
      Therefore, Helen is poor.
   b. All engineers need mathematics.
      Ron does not need mathematics.
      Therefore, Ron is not an engineer.
   c. All bicycles have tires.
      All tires use rubber.

18. Write each of the following in if-then form:
   a. All natural numbers are real numbers.
   b. Every circle is a closed figure.

19. Use De Morgan’s Law from Now Try This 1-16 to write a negation of each of the following:
   a. \( 3 + 5 \neq 9 \) and \( 3 \cdot 5 = 15 \).
   b. I am going or she is going.

---

Mathematical Connections 1-3

Communication

1. Explain why commands, questions, and opinions are not statements.
2. Explain how to write the negation of a quantified statement in the form “Some A are B.” Give an example.
3. Describe what is meant by a compound statement.
4. a. Describe under what conditions a disjunction is true.
    b. Describe under what conditions an implication is true.
5. What does the use of an “inclusive” or mean?
6. Explain how to determine if two statements are logically equivalent.
7. Describe Dr. No as completely as possible.

8. Consider the poem:

For want of a nail, the shoe was lost.
For want of a shoe, the horse was lost.
For want of a horse, the rider was lost.
For want of a rider, the battle was lost.
For want of a battle, the war was lost.
Therefore, for want of a nail, the war was lost.

a. Write each line as an if-then statement.
b. Does the conclusion follow logically? Why?

9. Most students today use Internet search engines such as Yahoo or Google. Efficient use of a search engine requires some knowledge of the connectives AND, OR, and NOT. One common type of advanced search is called a Boolean search (see Historical Note). With a Boolean search you can increase the accuracy of your search by specifying relationships among the keywords and phrases. The operator AND tells the search engine to search for all documents that contain both words, for example, “sports AND baseball.” Go online and explore the connectives AND, OR, and NOT. Explain your findings.

Open-Ended

10. Give two examples from mathematics for each of the following:
   a. A statement and its converse are true.
   b. A statement is true, but its converse is false.
   c. An “if, and only if,” true statement.
   d. An “if, and only if,” false statement.

Cooperative Learning

11. Each person in a group makes five sequences of statements similar to the ones in Example 1-13 but concerning mathematical objects, each with a valid or invalid conclusion. The statements should be as varied as possible. Each group member exchanges statements with another person—not revealing which conclusions are valid and which are not—and determines which of the other person’s conclusions are valid and which are not. The two group members compare their answers and discuss any discrepancies.

12. Discuss the paradox arising from the following:
   a. This textbook is 1000 pages long.
   b. The author of this textbook is Dante.
   c. The statements (a), (b), and (c) are all false.

Questions from the Classroom

13. A student says he does not understand the difference between \( \neg(p \land q) \) and \( \neg p \land q \). How do you explain this?
14. A student says that she does not see how a compound statement consisting of two simple sentences that are false can be true. How do you respond?
15. A student says that if the hypothesis is false, an argument cannot be valid. How do you respond?
Chapter Outline

I. Problem solving
   A. Problem solving can be guided by the following four-step process:
      1. Understanding the problem
      2. Devising a plan
      3. Carrying out the plan
      4. Looking back
   B. Important problem-solving strategies include the following:
      1. Look for a pattern.
      2. Make a table.
      3. Examine a simpler or special case of the problem to gain insight into the solution of the original problem.
      4. Identify a subgoal.
      5. Examine related problems and determine if the same technique can be applied.
      7. Write an equation.
      8. Draw a diagram.
      9. Guess and check.
     10. Use indirect reasoning.
     11. Use direct reasoning.
   C. Beware of mind-sets!

II. Mathematical patterns
   A. Patterns are an important part of problem solving.
   B. Patterns are used in inductive reasoning to form conjectures. Inductive reasoning is the method of making generalizations based on observations and patterns. A conjecture is a statement that is thought to be true but that has not yet been proved to be true or false. One way to prove a conjecture false is to produce a counterexample.
   C. A sequence is a group of terms in a definite order.
      1. Arithmetic sequence: Each successive term is obtained from the previous one by the addition of a fixed number called the difference. The nth term, \( a_n \), is given by \( a_n = a_1 + (n - 1)d \), where \( a_1 \) is the first term and \( d \) is the difference.
      2. Geometric sequence: Each successive term is obtained from its predecessor by multiplying it by a fixed, nonzero number called the ratio. The nth term, \( a_n \), is given by \( a_1 r^{n-1} \), where \( a_1 \) is the first term and \( r \) is the ratio.

III. Reasoning and logic
   A. A statement is a sentence that is either true or false but not both.
   B. The negation of a statement is a statement with the opposite truth value of the given statement. The negation of \( p \) is denoted by \( \neg p \).
   C. The compound statement \( p \land q \) is the conjunction of \( p \) and \( q \) and is defined to be true if, and only if, both \( p \) and \( q \) are true.
   D. The compound statement \( p \lor q \) is the disjunction of \( p \) and \( q \) and is true if either \( p \) or \( q \) are true.
   E. Statements of the form “if \( p \), then \( q \)” are conditionals or implications and are false only if \( p \) is true and \( q \) is false.
   F. Given the conditional \( p \rightarrow q \), the following can be found:
      1. Converse: \( q \rightarrow p \)
      2. Inverse: \( \neg p \rightarrow \neg q \)
      3. Contrapositive: \( \neg q \rightarrow \neg p \)
   G. If \( p \rightarrow q \) is true, the converse and the inverse are not necessarily true, but the contrapositive is true.
   H. Two statements are logically equivalent if, and only if, they have the same truth value. An implication and its contrapositive are logically equivalent.
   I. The statement “\( p \rightarrow q \) and \( q \rightarrow p \)” is written \( p \leftrightarrow q \), a biconditional, and referred to as “\( p \) if, and only if, \( q \)”.
   J. Laws to determine the validity of arguments include the law of detachment (modus ponens), modus tollens, and the chain rule.

5. In a recursive pattern, after one or more terms are given to start the sequence, each successive term of the sequence is obtained from previous term(s). The Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, 21, ..., is an example of a recursive sequence where \( F_1 = 1, F_2 = 1, F_{n+2} = F_n + F_{n+1} \).
   4. \( a^n = a \cdot a \cdot a \cdot a \cdot \ldots \cdot a \), where \( n \neq 0 \).
   5. \( a^0 = 1 \), where \( a \neq 0 \).
   6. Finding differences for a sequence is one technique for finding the next terms.

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Chapter Review

1. List three more terms that complete a pattern in each of the following:
   a. \(0, 1, 3, 6, 10, \ldots, \ldots, \ldots\)
   b. \(52, 47, 42, 37, \ldots, \ldots, \ldots\)
   c. \(6400, 3200, 1600, 800, \ldots, \ldots, \ldots\)
   d. \(1, 2, 3, 5, 8, 13, \ldots, \ldots, \ldots\)
   e. \(2, 5, 8, 11, 14, \ldots, \ldots, \ldots\)
   f. \(1, 4, 16, 64, \ldots, \ldots, \ldots\)
   g. \(0, 4, 8, 12, \ldots, \ldots, \ldots\)
   h. \(1, 8, 27, 64, \ldots, \ldots, \ldots\)

2. Classify each sequence in problem 1 as arithmetic, geometric, or neither.

3. Find a possible \(n\)th term in each of the following:
   a. 5, 8, 11, 14, \ldots
   b. 0, 7, 26, 63, \ldots
   c. 3, 9, 27, 81, 243, \ldots

4. Find the first five terms of the sequences whose \(n\)th term is given as follows:
   a. \(3n - 2\)
   b. \(n^2 + n\)
   c. \(4n - 1\)

5. Find the following sums:
   a. \(2 + 4 + 6 + 8 + 10 + \ldots + 200\)
   b. \(51 + 52 + 53 + 54 + \ldots + 151\)

6. Produce a counterexample, if possible, to disprove each of the following:
   a. If two odd numbers are added, then the sum is odd.
   b. If a number is odd, then it ends in a 1 or a 3.
   c. If two even numbers are added, then the sum is even.

7. Complete the following magic square; that is, complete the square so that the sum in each row, column, and diagonal is the same.

```
   16   3   2   13
  ---   ---   ---
    9    7   12
     4   14
```

8. How many people can be seated at 12 square tables lined up end to end if each table individually holds four persons?

9. A shirt and a tie sell for $9.50. The shirt costs $5.50 more than the tie. What is the cost of the tie?

10. If fence posts are to be placed in a row 5 m apart, how many posts are needed for 100 m of fence?

11. A total of 129 players entered a single-elimination handball tournament. In the first round of play, the top-seeded player received a bye and the remaining 128 players played in 64 matches. Thus, 65 players entered the second round of play. How many matches must be played to determine the tournament champion?

12. a. Use patterns to predict the next two lines.

\[
\begin{align*}
3 &= \frac{3 \cdot 2}{2} \\
3 + 6 &= \frac{6 \cdot 3}{2} \\
3 + 6 + 9 &= \frac{9 \cdot 4}{2} \\
3 + 6 + 9 + 12 &= \frac{12 \cdot 5}{2}
\end{align*}
\]

b. Show that this pattern works in general for adding consecutive multiples of 3.

13. If a complete turn of a car tire moves a car forward 6 ft, how many turns of the tire occur before the tire goes off its 50,000 mi warranty?

14. The members of Mrs. Grant’s class are standing in a circle; they are evenly spaced and are numbered in order. The student with number 7 is standing directly across from the student with number 17. How many students are in the class?

15. A carpenter has three large boxes. Inside each large box are two medium-sized boxes. Inside each medium-sized box are five small boxes. How many boxes are there altogether?

16. How many triangles are there in the following figure? Explain your reasoning.

17. Mary left her home and averaged 16 km/hr riding her bicycle on an uphill trip to Larry’s house. On the return trip over the same route, she averaged 20 km/hr. If it took 4 hr to make the return trip, how much cycling time did the entire trip take?

18. Use differences to find the next term in the following pattern:

\[5, 15, 37, 77, 141, \ldots\]
19. An ant farm can hold 100,000 ants. If the farm held 1500 ants on the first day, 3000 ants on the second day, 6000 ants on the third day, and so on, forming a geometric sequence, in how many days will the farm be full?

20. Toma's team entered a mathematics contest where teams of students compete by answering questions that are worth either 3 points or 5 points. No partial credit was given. Toma's team scored 44 points on 12 questions. How many 5-point questions did the team answer correctly?

21. Three pieces of wood are needed for a project. They are to be cut from a 90-cm-long piece of wood. The longest piece is to be 3 times as long as the middle-sized piece and the shortest piece is to be 10 cm shorter than the middle-sized piece. Can this be done with two cuts? If so, tell how.

22. I am thinking of a number. If I double it, square the result, then divide by 2 and add 8, I get 40. What is my number?

23. Explain the difference between the following two statements: (i) All students passed the final. (ii) Some students passed the final.

24. Which of the following are statements?
   a. The moon is inhabited.
   b. \(3 + 5 = 8\).
   c. \(x + 7 = 15\).
   d. Some women have Ph.D.'s in mathematics.

25. Negate each of the following:
   a. Some women smoke.
   b. \(3 + 5 = 8\).
   c. All heavy-metal rock is loud.
   d. Beethoven wrote only classical music.

26. Write the converse, inverse, and contrapositive of the following: If we have a rock concert, someone will faint.

*27. Use truth tables to show that \(p \rightarrow \sim q = q \rightarrow \sim p\).

*28. Construct truth tables for each of the following:
   a. \((p \land \sim q) \lor (p \land q)\)
   b. \([(p \lor q) \land \sim p] \rightarrow q\)

*29. Find valid conclusions for the following hypotheses:
   a. All Americans love Mom and apple pie.
   Joe Czernyu is an American.
   b. Steel eventually rusts.
   The Statue of Liberty has a steel structure.
   c. Albertina passed Math 100 or Albertina dropped out.
   Albertina did not drop out.

*30. Write the following argument symbolically and then determine its validity:
   If you are fair-skinned, you will sunburn.
   If you sunburn, you will not go to the dance.
   If you do not go to the dance, your parents will want to know why you didn’t go to the dance.
   Therefore, you are not fair-skinned.

*31. State whether the conclusion is valid in each case and tell why.
   a. If Bob scores at least 80 on the final, he will pass the course.
   Bob did not pass the course.
   Therefore, Bob did not score at least 80 on the final.
   b. If you build it, they will come.
   You build it.
   Therefore, they will come.

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**Selected Bibliography**


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Hylton-Lindsay, A. “Problem-Solving, Patterns, Probability, Pascal, and Palindromes.” *Mathematics Teaching in the Middle School* 8 (February 2003): 288–293.


