CHAPTER 3

Whole Numbers and Their Operations

Preliminary Problem

Using exactly five 5s and only addition, subtraction, multiplication, and division, write an expression that equals each of the numbers from 1 to 10. You do not have to use all operations. Numbers such as 55 are permitted; for example, 5 could be written as $5 + [(5 - 5) \cdot 55]$. 

In Section 2-1 we saw that the concept of one-to-one correspondence between sets can be used to introduce children to the concept of a number. In NCTM’s *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics*, we find the following:

Children develop an understanding of the meanings of whole numbers and recognize the number of objects in small groups without counting and by counting—the first and most basic mathematical algorithm. They understand that number words refer to quantity. They use one-to-one correspondence to solve problems by matching sets and comparing number amounts and in counting objects to 10 and beyond. They understand that the last word that they state in counting tells “how many,” they count to determine number amounts and compare quantities (using language such as “more than” and “less than”), and they order sets by the number of objects in them. (p. 11)

In the following *Peanuts* cartoon, it seems that Lucy’s little brother has not yet learned to associate number words with a collection of objects. He will soon learn that this set of fingers can be put into one-to-one correspondence with many sets of objects that can be counted. He will associate the word *three* not only with Lucy’s three upheld fingers but with other sets of objects with this same cardinal number.

In this chapter, we will investigate operations involving whole numbers. As stated in the *Principles and Standards (PSSM)* in “Number and Operation” for grades pre-K–2, all students on this level should understand meanings of operations and how they relate to one another. In particular, *PSSM* states that in pre-K–2 all students should:

- understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations;
- understand the effects of adding and subtracting whole numbers;
- understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally. (p. 78)

### 3-1 Addition and Subtraction of Whole Numbers

When zero is included with the set of natural numbers, \( N = \{1, 2, 3, 4, 5, \ldots \} \), we have the set of whole numbers, denoted \( W = \{0, 1, 2, 3, 4, 5, \ldots \} \). In this section, we provide a variety of models for teaching computational skills involving whole numbers and allow you to revisit mathematics for the deeper understanding that teachers need.
Addition of Whole Numbers

Children encounter addition in preschool years by combining objects and wanting to know how many objects there are in the combined set. They may “count on” as suggested by Carpenter and Moser in the Research Note, or they may count the objects to find the cardinal number of the combined set.

Set Model

A set model is one way to represent addition of whole numbers. Suppose Jane has 4 blocks in one pile and 3 in another. If she combines the two groups, how many objects are there in the combined group? Figure 3-1 shows the solution as it might appear in an elementary school text. The combined set of blocks is the union of the disjoint sets of 4 blocks and 3 blocks. After the sets have been combined, some children count the objects to determine that there are 7 in all. Note the importance of the sets being disjoint or having no elements in common. If the sets have common elements, then an incorrect conclusion can be drawn.

Using set terminology, we define addition formally.

**Definition of Addition of Whole Numbers**

Let \( A \) and \( B \) be two disjoint finite sets. If \( n(A) = a \) and \( n(B) = b \), then \( a + b = n(A \cup B) \).

The numbers \( a \) and \( b \) in \( a + b \) are the addends and \( a + b \) is the sum.

**NOW TRY THIS 3-1** If the sets in the preceding definition of addition of whole numbers are not disjoint, explain why the definition is incorrect.

Historical Note

Historians think that the word *zero* originated from the Hindu word *śūnya*, which means “void.” Then *śūnya* was translated into the Arabic *ṣifr*, which when translated to Latin became *zeuppīrnum*, from which the word *zero* was derived.
Number-Line (Measurement) Model

The set model for addition may not be the best model for some additions. For example, consider the following questions:

1. Josh has 4 feet of red ribbon and 3 feet of white ribbon. How many feet of ribbon does he have altogether?
2. One day, Gail drank 4 ounces of orange juice in the morning and 3 ounces at lunchtime. If she drank no other orange juice that day, how many ounces of orange juice did she drink for the entire day?

A number line can be used to model whole-number addition and answer questions 1 and 2. Any line marked with two fundamental points, one representing 0 and the other representing 1, can be turned into a number line. The points representing 0 and 1 mark the ends of a unit segment. Other points can be marked and labeled, as shown in Figure 3-2. Any two consecutive whole numbers on the number line in Figure 3-2 mark the ends of a segment that has the same length as the unit segment.

Addition problems may be modeled using directed arrows (vectors) on the number line. For example, the sum of 4 + 3 is shown in Figure 3-2. Arrows representing the addends, 4 and 3, are combined into one arrow representing the sum 4 + 3. Figure 3-2 poses an inherent problem for students. If an arrow starting at 0 and ending at 3 represents 3, why should an arrow starting at 4 and ending at 7 represent 3? Students need to understand that the sum represented by any two directed arrows can be found by placing the endpoint of the first directed arrow at 0 and then joining to it the directed arrow for the second number with no gaps or overlaps. The sum of the numbers can then be read. We have depicted the addends as arrows (or vectors) above the number line, but students typically concatenate (connect) the arrows directly on the line.

NOW TRY THIS 3-2 A common error is that students represent 3 as an arrow on the number line sometimes starting at 1, as shown in Figure 3-3. Explain why this is not appropriate.

The symbol “+” first appeared in a 1417 manuscript and was a short way of writing the Latin word *et*, which means “and.” The word *minus* means “less” in Latin. First written as an *m*, it was later shortened to a horizontal bar. Johannes Widman wrote a book in 1489 that made use of the + and − symbols for addition and subtraction.
Ordering Whole Numbers

In the NCTM grade 1 Curriculum Focal Points, we find the following:

Children compare and order whole numbers (at least to 100) to develop an understanding of and solve problems involving the relative sizes of these numbers. They think of whole numbers between 10 and 100 in terms of groups of tens and ones (especially recognizing the numbers 11 to 19 as 1 group of ten and particular numbers of ones). They understand the sequential order of the counting numbers and their relative magnitudes and represent numbers on a number line. (p. 13)

In Chapter 2, we used the concept of a set and the concept of a one-to-one correspondence to define greater-than relations. A horizontal number line can also be used to describe greater-than and less-than relations in the set of whole numbers. For example, in Figure 3-2, notice that 4 is to the left of 7 on the number line. We say, “four is less than seven,” and we write $4 < 7$. We can also say “seven is greater than four” and write $7 > 4$. Since 4 is to the left of 7, there is a natural number that can be added to 4 to get 7, namely, 3. Thus, $4 < 7$ because $4 + 3 = 7$. We can generalize this discussion to form the following definition of less than.

**Definition of Less Than**

For any whole numbers $a$ and $b$, $a$ is less than $b$, written $a < b$, if, and only if, there exists a natural number $k$ such that $a + k = b$.

Sometimes equality is combined with the inequalities, greater than and less than, to give the relations greater than or equal to and less than or equal to, denoted $\geq$ and $\leq$. Thus, $a \leq b$ means $a < b$ or $a = b$. The emphasis with respect to these symbols is on the or, so $3 \leq 5$, $5 \leq 3$, and $3 \leq 3$ are all true statements.

Whole-Number Addition Properties

Any time two whole numbers are added, we are guaranteed that a unique whole number will be obtained. This property is sometimes referred to as the closure property of addition of whole numbers. We say that “the set of whole numbers is closed under addition.”

**Theorem 3–1: Closure Property of Addition of Whole Numbers**

If $a$ and $b$ are whole numbers, then $a + b$ is a whole number.

**Remark** The closure property implies that the sum of two whole numbers exists and that the sum is a unique whole number; for example, $5 + 2$ is a unique whole number and we identify that number as 7.

**Now Try This 3–3** Determine whether each of the following sets is closed under addition:

a. $E = \{2, 4, 6, 8, 10, \ldots \}$

b. $F = \{1, 3, 5, 7, 9\}$
In *Principles and Standards* we find the following:

In developing the meaning of addition and subtraction with whole numbers, students should also encounter the properties of operations, such as the commutativity and associativity of addition. Although some students discover and use properties of operations naturally, teachers can bring these properties to the forefront through class discussions. (p. 83)

Figure 3-4(a) shows two additions. Pictured above the number line is and below the number line is . The sums are the same. Figure 3-4(b) shows the same sums obtained with colored rods and the result being the same. Both illustrations in Figure 3-4 demonstrate the idea that two whole numbers can be added in either order. This property is true in general and is the commutative property of addition of whole numbers. We say that “addition of whole numbers is commutative.” The word *commutative* is derived from *commute*, which means “to interchange.”

- **Figure 3-4**

**Theorem 3–2: Commutative Property of Addition of Whole Numbers**

If and are any whole numbers, then .

The commutative property of addition of whole numbers is not obvious to many young children. They may be able to find the sum and not be able to find the sum . Using *counting on*, can be computed by starting at 9 and then counting on two more as “ten” and “eleven.” To compute without the commutative property, the *counting on* is more involved. Students need to understand that is another name for .

**NOW TRY THIS 3-4** Use the set model to show the commutative property for .

Another property of addition is demonstrated when we select the order in which to add three or more numbers. For example, we could compute by grouping the 24 and the 8 together: . (The parentheses indicate that the first two numbers are grouped together.) We might also recognize that it is easy to add any number to 10 and compute it as . This example illustrates the associative property of addition of whole numbers. The word *associative* is derived from the word *associate*, which means “to unite.”
Whole Numbers and Their Operations

Another property of addition of whole numbers is seen when one addend is 0. In Figure 3-5, set $A$ has 5 blocks and set $B$ has 0 blocks. The union of sets $A$ and $B$ has only 5 blocks.

$n(A) = 5$
$n(B) = 0$ because $B = \emptyset$

$n(A) + n(B) = 5 + 0 = 5 = n(A \cup B)$

This example uses the set model to illustrate the following property of whole numbers:

**Theorem 3–4: Identity Property of Addition of Whole Numbers**

There is a unique whole number 0, the **additive identity**, such that for any whole number $a$, $a + 0 = a = 0 + a$.

Notice how the associative and identity properties are introduced on the grade 3 student page on p. 117. Work through parts 4–7.

**Example 3-1**

Which properties justify each of the following?

a. $5 + 7 = 7 + 5$

b. $1001 + 733$ is a unique whole number.

c. $(3 + 5) + 7 = (5 + 3) + 7$

d. $(8 + 5) + 2 = 2 + (8 + 5) = (2 + 8) + 5$

**Solution**

a. Commutative property of addition

b. Closure property of addition

c. Commutative property of addition

d. Commutative and associative properties of addition
What is the Associative Property?

The **Associative (grouping) Property of Addition** says that you can group addends in any way and the sum will be the same.

\[
\begin{align*}
(3 + 1) + 2 &= 6 \\
3 + (1 + 2) &= 6 \\
\end{align*}
\]

So \((3 + 1) + 2 = 3 + (1 + 2)\).

**Talk About It**

4. Evan says, “You can rewrite \((5 + 3) + 2\) as \(8 + 2\).” Do you agree? Explain.

What is the Identity Property?

The **Identity (zero) Property of Addition** says that the sum of any number and zero is that same number.

\[
\begin{align*}
6 + 0 &= 6 \\
\end{align*}
\]

**Talk About It**

5. How could you use the Identity Property of Addition to find \(536 + 0\)?

**CHECK**

Find each sum.

1. \(2 + (5 + 3)\)  
2. \(3 + (1 + 4)\)  
3. \(3 + 2 + 6\)

Write each missing number.

4. \(5 + 2 = 2 + \_\)  
5. \(\_ + 4 = 4\)  
6. \((1 + 4) + 3 = \_ + (4 + 3)\)

7. **Number Sense** What property of addition is shown in the following number sentence? Explain. \(4 + (5 + 2) = (5 + 2) + 4\)

Source: Scott Foresman-Addison Wesley Math, Grade 3, 2008 (p. 67).
Mastering Basic Addition Facts

Certain mathematical facts are basic addition facts. Basic addition facts are those involving a single digit plus a single digit. In the Dennis The Menace cartoon, it seems that Dennis has not mastered the basic addition facts.

One method of learning the basic facts is to organize them according to different derived fact strategies, as suggested in the Research Note. Several strategies are listed below.

1. **Counting on.** The strategy of counting on from the greater of the addends can be used any time we need to add whole numbers, but it is inefficient. It is usually used when one addend is 1, 2, or 3. For example, in the cartoon Dennis could have computed $4 + 2$ by starting at 4 and then counting on 5, 6. Likewise, $3 + 3$ would be computed by starting at 3 and then counting on 4, 5, 6.

2. **Doubles.** The next strategy involves the use of doubles. Doubles such as $3 + 3$ in the cartoon receive special attention. After students master doubles, $doubles + 1$ and $doubles + 2$ can be learned easily. For example, if a student knows $6 + 6 = 12$, then $6 + 7$ is $(6 + 6) + 1$, or 1 more than the double of 6, or 13. Likewise, $7 + 9$ is $(7 + 7) + 2$, or 2 more than the double of 7, or 16.

3. **Making 10.** Another strategy is that of making 10 and then adding any leftover. For example, we could think of $8 + 5$ as shown in Figure 3-6. Notice that we are really using the associative property of addition.

![Figure 3-6](image-url)
4. Counting back. The strategy of counting back is usually used when one number is 1 or 2 less than 10. For example, because 9 is 1 less than 10, then \( 9 + 7 \) is 1 less than \( 10 + 7 \), or 16. In symbols, this is \( 9 + 7 = (10 + 7) - 1 = 17 - 1 = 16 \). Also, \( 8 + 7 = (10 + 7) - 2 = 17 - 2 = 15 \).

Many basic facts might be classified under more than one strategy. For example, we could find \( 9 + 8 \) using *making 10* as \( 9 + (1 + 7) = (9 + 1) + 7 = 10 + 7 = 17 \) or using a *double + 1* as \( (8 + 8) + 1 \).

**Subtraction of Whole Numbers**

In the grade 1 *Focal Points* we find the following:

> By comparing a variety of solution strategies, children relate addition and subtraction as inverse operations. (p. 13)

In elementary school, operations that “undo” each other are inverse operations. Subtraction is the inverse operation for addition. It is sometimes hard for students to understand this inverse relationship between the two operations, as the cartoon demonstrates.

**B.C.**

In *Principles and Standards* we find the following:

> Students develop further understandings of addition when they solve missing-addend problems that arise from stories or real situations. Further understandings of subtraction are conveyed by situations in which two collections need to be made equal or one collection needs to be made a desired size. Some problems, such as “Carlos had three cookies. Maria gave him some more, and now he has eight. How many did she give him?” can help students see the relationship between addition and subtraction. (p. 83)

We can model subtraction of whole numbers using several solution strategies, including the *take-away* model, the *missing-addend* model, the *comparison* model, and the *number-line (measurement)* model.
Take-Away Model

In addition, we imagine a second set of objects as being joined to a first set, but in subtraction, we imagine a second set as being taken away from a first set. For example, suppose we have 8 blocks and take away 3 of them. We illustrate this in Figure 3-7 and record this process as \(8 - 3 = 5\).

![Figure 3-7](image)

Now Try This 3-5 Recall how addition of whole numbers was defined using the concept of union of two disjoint sets. Similarly, write a definition of subtraction of whole numbers using the concepts of subsets and set difference.

Missing-Addend Model

A second model for subtraction, the missing-addend model, relates subtraction and addition. In Figure 3-7, \(8 - 3\) is pictured as 8 blocks “take away” 3 blocks. The number of blocks left is the number \(8 - 3\), or 5. This can also be thought of as the number of blocks that could be added to 3 blocks in order to get 8 blocks, that is,

\[
3 + \text{?} = 8
\]

The number \(8 - 3\), or 5, is the missing addend in the equation

\[
3 + \text{?} = 8.
\]

We can also relate the missing-addend approach to sets or to a number line. The subtraction \(8 - 3\) is illustrated in Figure 3-8(a) using sets and in Figure 3-8(b) using the number line.

![Figure 3-8](image)

The missing-addend model gives elementary school students an opportunity to begin algebraic thinking. An unknown is a major part of the problem of trying to decide the difference of \(8 - 3\).
Refer to the student page on p. 122 to see how grade 3 students are shown how addition and subtraction are related using a fact family. Answer the Talk About It questions on the bottom of the student page.

Cashiers often use the missing-addend model. For example, if the bill for a movie is $8 and you pay $10, the cashier might calculate the change by saying “8 and 2 is 10.” This idea can be generalized: For any whole numbers \( a \) and \( b \) such that \( a \geq b \), \( a - b \) is the unique whole number such that \( b + \Box = a \). That is, \( a - b \) is the unique solution of the equation \( b + \Box = a \). The definition can be written as follows:

**Definition of Subtraction of Whole Numbers**

For any whole numbers \( a \) and \( b \) such that \( a \geq b \), \( a - b \) is the unique whole number \( c \) such that \( b + c = a \).

### Comparison Model

Another way to consider subtraction is by using a comparison model. Suppose Juan has 8 blocks and Susan has 3 blocks and we want to know how many more blocks Juan has than Susan. We can pair Susan’s blocks with some of Juan’s blocks, as shown in Figure 3-9, and determine that Juan has 5 more blocks than Susan. We also write this as \( 8 - 3 = 5 \).

![Figure 3-9](Juan's 8 blocks - Susan's 3 blocks = Difference of 5 blocks)

### Number-Line (Measurement) Model

Subtraction of whole numbers can also be modeled on a number line using directed arrows, as suggested in Figure 3-10, which shows that \( 5 - 3 = 2 \).

![Figure 3-10](Number Line 0 to 7 with 5 - 3 = 2)

The following four problems illustrate each of the four models for subtraction. In all four problems the answer is 5 but each can be thought of using a different model.

1. **Take-away model.** Al had $9 and spent $4. How much did he have left?
2. **Missing-addend model.** Al has read 4 chapters of a 9-chapter book. How many chapters does he have left to read?
3. **Comparison model.** Al has 9 books and Betty has 4 books. How many more books does Al have than Betty?
4. **Number-line model.** Al completed a 9-mile hike in two days. He hiked 4 miles on the second day. How far did he hike on the first day?
Relating Addition and Subtraction

How are addition and subtraction related?

You can think about parts and the whole to show how addition and subtraction are related.

You can write a fact family when you know the parts and the whole.

Fact family:

- $5 + 8 = 13$
- $13 - 8 = 5$
- $8 + 5 = 13$
- $13 - 5 = 8$

**Example**

Find $12 - 7$.

<table>
<thead>
<tr>
<th>What You Think</th>
<th>What You Write</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7 + ? = 12$</td>
<td>$12 - 7 = 5$</td>
</tr>
<tr>
<td>$7 + 5 = 12$</td>
<td></td>
</tr>
</tbody>
</table>

**Talk About It**

1. What are the other three number sentences in the fact family with $6 + 3 = 9$?

2. What addition fact can help you find $11 - 3$?
Properties of Subtraction

In an attempt to find $3 - 5$, we use the definition of subtraction: $3 - 5 = c$ if, and only if, $c + 5 = 3$. Since there is no whole number $c$ that satisfies the equation, $3 - 5$ is not meaningful in the set of whole numbers. In general, it can be shown that if $a < b$, then $a - b$ is not meaningful in the set of whole numbers. Therefore, subtraction is not closed on the set of whole numbers.

NOW TRY THIS 3-6 Which of the following properties hold for subtraction of whole numbers?

Explain.

a. Closure property
b. Associative property
c. Commutative property
d. Identity property

Introductory Algebra Using Whole-Number Addition and Subtraction

Sentences such as $9 + 5 = x$ and $12 - y = 4$ can be true or false depending on the values of $x$ and $y$. For example, if $x = 10$, then $9 + 5 = x$ is false. If $y = 8$, then $12 - y = 4$ is true. If the value that is used makes the equation true, it is a solution to the equation.

NOW TRY THIS 3-7 Find the solution for each of the following where $x$ is a whole number:

a. $x + 8 = 13$

b. $15 - x = 8$

c. $x > 9$ and $x < 11$

Assessment 3-1A

1. Give an example to show why, in the definition of addition, sets $A$ and $B$ must be disjoint.

2. For which of the following is it true that $n(A) + n(B) = n(A \cup B)$?
   a. $A = \{a, b, c\}$, $B = \{d, e\}$
   b. $A = \{a, b, c\}$, $B = \{b, c\}$
   c. $A = \{a, b, c\}$, $B = \emptyset$

3. If $n(A) = 3$, $n(B) = 5$, and $n(A \cup B) = 6$, what do you know about $n(A \cap B)$?

4. If $n(A) = 3$ and $n(A \cup B) = 6$,
   a. what are the possible values for $n(B)$?
   b. If $A \cap B = \emptyset$, what are the possible values of $n(B)$?

5. Explain whether the following given sets are closed under addition:
   a. $B = \{0\}$
   b. $T = \{0, 3, 6, 9, 12, \ldots\}$
   c. $N = \{1, 2, 3, 4, 5, \ldots\}$
   d. $V = \{3, 5, 7\}$
   e. $\{x \mid x \in W \text{ and } x > 10\}$

6. Each of the following is an example of one of the properties for addition of whole numbers. Identify the property illustrated.
   a. $6 + 3 = 3 + 6$
   b. $(6 + 3) + 5 = 6 + (3 + 5)$
   c. $(6 + 3) + 5 = (3 + 6) + 5$
   d. $5 + 0 = 0 + 5$
   e. $5 + 0 = 0 + 5$
   f. $(a + c) + d = a + (c + d)$

7. In the definition of less than, can the natural number $k$ be replaced by the whole number $k$? Why or why not?

8. a. Recall how we have defined less-than and greater-than relations, and give a similar definition using the concept of subtraction for each of the following:
   i. $a < b$
   ii. $a > b$
   b. Use subtraction to define $a \geq b$. 

9. Find the next three terms in each of the following arithmetic sequences:
   a. 8, 13, 18, 23, 28, ____, ____, ____
   b. 98, 91, 84, 77, 70, 63, ____, ____, ____

10. If \( A, B, \) and \( C \) each stand for a different single digit from 1 to 9, answer the following if
    \[ A + B = C \]
    a. What is the greatest digit that \( C \) could be? Why?
    b. What is the greatest digit that \( A \) could be? Why?
    c. What is the smallest digit that \( C \) could be? Why?
    d. If \( A, B, \) and \( C \) are even, what number(s) could \( C \) be? Why?
    e. If \( C \) is 5 more than \( A \), what number(s) could \( B \) be? Why?
    f. If \( A \) is 3 times \( B \), what number(s) could \( C \) be? Why?

11. Assuming the same pattern continues, find the total of the terms in the 50th row in the following figure:
    
    |        | 1st row         |         | 2nd row         |         | 3rd row         |
    |--------|----------------|--------|----------------|--------|----------------|
    | 1      | 1              | 1 - 1  | 1 - 1 + 1      | 1 - 1 + 1 - 1 | 1 - 1 + 1 - 1 + 1 |
    | 5      | 7              | 10     | 14             | 13     | 18             |

12. Make each of the following a magic square (a magic square was defined in Chapter 1):
    a. 
    
    | 1 | 6 |
    |---|---|
    | 5 | 7 |
    |---|---|
    | 4 | 2 |

    b. 
    
    | 17 | 10 |
    |---|---|
    | 13 | 18 |

13. a. At a volleyball game, the players stood in a row ordered by height. If Kent is shorter than Mischa, Sally is taller than Mischa, and Vera is taller than Sally, who is the tallest and who is the shortest?
   b. Write possible heights for the players in part (a).

14. Rewrite each of the following subtraction problems as an equivalent addition problem:
   a. \( 9 - 7 = x \)
   b. \( x - 6 = 3 \)
   c. \( 9 - x = 2 \)

15. Refer to the student page on page 122 to recall the description of a fact family.
   a. Write the fact family for \( 8 + 3 = 11 \).
   b. Write the fact family for \( 13 - 8 = 5 \).

16. What conditions, if any, must be placed on \( a, b, \) and \( c \) in each of the following to make sure that the result is a whole number?
   a. \( a - b \)
   b. \( a - (b - c) \)

17. Illustrate \( 8 - 5 = 3 \) using each of the following models:
   a. Take-away
   b. Missing addend
   c. Comparison
   d. Number line

18. Find the solution for each of the following:
   a. \( 3 + (4 + 7) = (3 + x) + 7 \)
   b. \( 8 + 0 = x \)
   c. \( 5 + 8 = 8 + x \)
   d. \( x + 8 = 12 + 5 \)

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### Assessment 3-IB

1. For which of the following is it true that \( n(A) + n(B) = n(A \cup B) \)?
   a. \( A = \{a, b\}, B = \{d, e\} \)
   b. \( A = \{a, b, c\}, B = \{b, c, d\} \)
   c. \( A = \{a\}, B = \emptyset \)

2. If \( n(A) = 3, n(B) = 5, \) and \( n(A \cap B) = 1, \) what do you know about \( n(A \cup B) \)?

3. Explain whether the following given sets are closed under addition:
   a. \( B = \{0, 1\} \)
   b. \( T = \{0, 4, 8, 12, 16, \ldots\} \)
   c. \( F = \{5, 6, 7, 8, 9, 10, \ldots\} \)
   d. \( \{x \mid x \in W \text{ and } x > 100\}\)

4. Set \( A \) contains the element 1. What other whole numbers must be in set \( A \) for it to be closed under addition?

5. Set \( A \) is closed under addition and contains the numbers 2, 5, and 8. List six other elements that must be in \( A \).

6. Each of the following is an example of one of the properties of whole-number addition. Fill in the blank to make a true statement and identify the property.
   a. \( 3 + 4 = \_\_\_ + 3 \)
   b. \( 5 + (4 + 3) = (4 + 3) + \_\_\_ \)
   c. \( 8 + \_\_\_ = 8 \)
   d. \( 3 + (4 + 5) = (3 + \_\_\_) + 5 \)
   e. \( 3 + 4 \) is a unique \_\_\_ number.

7. Each of the following is an example of one of the properties for addition of whole numbers. Identify the property illustrated.
   a. \( 6 + 8 = 8 + 6 \)
   b. \( (6 + 3) + 0 = 6 + 3 \)
c. \((6 + 8) + 2 = (8 + 6) + 2\)
d. \((5 + 3) + 2 = 5 + (3 + 2)\)

8. Find the next three terms in each of the following arithmetic sequences:
   a. \(5, 12, 19, 26, 33, ____, ____, ____\)
   b. \(63, 59, 55, 51, 47, ____, ____, ____\)

9. If \(A, B, C,\) and \(D\) each stand for a different single digit from 1 to 9, answer each of the following if
   \[
   \begin{align*}
   A + B &= CD \\
   \end{align*}
   \]
   a. What is the value of \(C\)? Why?
b. Can \(D\) be 1? Why?
c. If \(D\) is 7, what values can \(A\) be?
d. If \(A\) is 6 greater than \(B\), what is the value of \(D\)?

10. a. A domino set contains all number pairs from double-0 to double-6, with each number pair occurring only once; for example, the following domino counts as 2-4 and 4-2. How many dominoes are in the set?
   \[
   \begin{array}{c}
   \text{3} \\
   \text{1} \\
   \text{2} \\
   \text{5} \\
   \end{array}
   \]
   b. When considering the sum of all dots on a single domino in an ordinary set of dominoes, explain how the commutative property might be important.

11. Rewrite each of the following subtraction problems as an equivalent addition problem:
   a. \(9 - 3 = x\)

12. Refer to the student page on page 122 to recall the description of a fact family.
   a. Write the fact family for \(9 + 8 = 17\).
   b. Write the fact family for \(15 - 7 = 8\).

13. Show that each of the following is true. Give a property of addition to justify each step.
   a. \(a + (b + c) = c + (a + b)\)
   b. \(a + (b + c) = (c + b) + a\)

14. Illustrate \(7 - 3 = 4\) using each of the following models:
   a. Take-away
   b. Missing addend
   c. Comparison
   d. Number line

15. Find the solution for each of the following:
   a. \(12 - x = x + 6\)
   b. \((9 - x) - 6 = 1\)
   c. \(3 + x = x + 3\)
   d. \(15 - x = x - 7\)
   e. \(14 - x = 7 - x\)

16. Rob has 11 pencils. Kelly has 5 pencils. Which number sentence shows how many more pencils Rob has than Kelly?
   (i) \(11 + 5 = 16\)
   (ii) \(16 - 5 = 11\)
   (iii) \(11 - 5 = 6\)
   (iv) \(11 - 6 = 5\)

### Mathematical Connections 3-1

**Communication**

1. In a survey of 52 students, 22 said they were taking algebra and 30 said they were taking biology. Are there necessarily 52 students taking algebra or biology? Why?
2. To find \(9 + 7\) a student says she thinks of \(9 + (1 + 6) = (9 + 1) + 6 = 10 + 6 = 16\). What property or properties is she using?
3. In Figure 3-2, arrows were used to represent numbers in completing an addition. Explain whether you think an arrow starting at 0 and ending at 3 represents the same number as an arrow starting at 4 and ending at 7. How would you explain this to students?
4. When subtraction and addition appear in an expression without parentheses, it is agreed that the operations are performed in order of their appearance from left to right. Taking this into account, answer the following:
   a. Use an appropriate model for subtraction to explain why
      \[a - b - c = a - c - b\]
      assuming that all expressions are meaningful.
   b. Use an appropriate model for subtraction to explain why
      \[a - b - c = a - (b + c)\]

5. Explain whether it is important for elementary students to learn more than one model for performing the operations of addition and subtraction.

6. Do elementary students still have to learn their basic facts when the calculator is part of the curriculum? Why or why not?

7. Explain how the model shown can be used to illustrate each of the following addition and subtraction facts:
   a. \(9 + 4 = 13\)
   b. \(4 + 9 = 13\)
   c. \(4 = 13 - 9\)
   d. \(9 = 13 - 4\)

| 9 | 4 | 13 |
8. How are addition and subtraction related? Explain.
9. Why is 0 not an identity for subtraction? Explain.

Open-Ended
10. Describe any model not in this text that you might use to teach addition to students.
11. Suppose \( A \subseteq B \). If \( n(A) = a \) and \( n(B) = b \), then \( b - a \) could be defined as \( n(B - A) \). Choose two sets \( A \) and \( B \) and illustrate this definition.
12. a. Create a word problem in which the set model would be more appropriate to show \( 25 + 8 = 33 \).
   b. Create a word problem in which the number-line (measurement) model would be more appropriate to show \( 25 + 8 = 33 \).

Cooperative Learning
13. Discuss with your group each of the following. Use the basic addition fact table for whole numbers shown.

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
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<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
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<td>6</td>
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<td>12</td>
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<td>7</td>
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<td>11</td>
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<td>15</td>
<td>16</td>
<td>17</td>
<td>17</td>
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<tr>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>

a. How does the table show the closure property?
   b. How does the table show the commutative property?
   c. How does the table show the identity property?
   d. How do the addition properties help students learn their basic facts?

14. Suppose that a number system used only four symbols, \( a, b, c, \) and \( d \), and the operation \( \Delta \); and the system operated as shown in the table. Discuss with your group each of the following:

<table>
<thead>
<tr>
<th>( \Delta )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
<td>( d )</td>
</tr>
<tr>
<td>( b )</td>
<td>( b )</td>
<td>( c )</td>
<td>( d )</td>
<td>( a )</td>
</tr>
<tr>
<td>( c )</td>
<td>( c )</td>
<td>( d )</td>
<td>( a )</td>
<td>( b )</td>
</tr>
<tr>
<td>( d )</td>
<td>( d )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
</tr>
</tbody>
</table>

a. Is the system closed? Why?
b. Is the operation commutative? Why?
c. Does the operation have an identity? If so, what is it?
d. Try several examples to investigate the associative property of this operation.

15. Have each person in your group choose a different grade textbook and report on when and how subtraction of whole numbers is introduced. Compare with different ways subtraction is introduced in this section.

Questions from the Classroom
16. A student says that 0 is the identity for subtraction. How do you respond?
17. A student claims that on the following number line, the arrow doesn’t really represent 3 because the end of the arrow does not start at 0. How do you respond?

18. A student asks why we use subtraction to determine how many more pencils Sam has than Karly if nothing is being taken away. How do you respond?

19. A student claims that subtraction is closed with respect to the whole numbers. To show this is true, she shows that \( 8 - 5 = 3, 5 - 2 = 3, 6 - 1 = 5, \) and \( 12 - 7 = 5 \) and says she can show examples like this all day that yield whole numbers when the subtraction is performed. How do you respond?

20. John claims that he can get the same answer to the problem below by adding up (begin with \( 4 + 7 \)) or by adding down (begin with \( 8 + 7 \)). He wants to know why and if this works all the time. How do you respond?

\[
\begin{array}{c}
8 \\
\hline
7 \\
\hline
\end{array}
\]

Third International Mathematics and Science Study (TIMSS) Questions

Ali had 50 apples. He sold some and then had 20 left. Which of these is a number sentence that shows this?

a. \( \square - 20 = 50 \)
b. \( 20 - \square = 50 \)
c. \( \square - 50 = 20 \)
d. \( 50 - \square = 20 \)

TIMSS 2003, Grade 4

The rule for the table is that numbers in each row and column must add up to the same number. What number goes in the center of the table?

a. 1
b. 2
c. 7
d. 12

TIMSS 2003, Grade 4

\[
\begin{array}{ccc}
4 & 11 & 6 \\
\hline
9 & 5 & \\
\hline
8 & 3 & 10 \\
\end{array}
\]
BRAIN TEASER  Use Figure 3-11 to design an unmagic square. That is, use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 exactly once so that every column, row, and diagonal adds to a different sum.

3-2  Algorithms for Whole-Number Addition and Subtraction

In the grade 2 *Curriculum Focal Points*, we find the following regarding fluency with multi-digit addition and subtraction:

Children use their understanding of addition to develop quick recall of basic addition facts and related subtraction facts. They solve arithmetic problems by applying their understanding of models of addition and subtraction (such as combining or separating sets or using number lines), relationships and properties of number (such as place value), and properties of addition (commutativity and associativity). Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multidigit whole numbers. They select and apply appropriate methods to estimate sums and differences or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. (p. 14)

In *Principles and Standards* we also find a discussion of computational fluency and standard algorithms.

By the end of grade 2, students should know the basic addition and subtraction combinations, should be fluent in adding two-digit numbers, and should have methods for subtracting two-digit numbers. At the grades 3–5 level, as students develop the basic number combinations for multiplication and division, they should also develop reliable algorithms to solve arithmetic problems efficiently and accurately. These methods should be applied to larger numbers and practiced for fluency . . . students must become fluent in arithmetic computation—they must have efficient and accurate methods that are supported by an understanding of numbers and operations. “Standard” algorithms for arithmetic computation are one means of achieving this fluency. (p. 35)

The previous section introduced the operations of addition and subtraction of whole numbers, and now, as pointed out in the *Focal Points* and the *Principles and Standards*, it is time to focus on computational fluency—having and using efficient and accurate methods for computing. *PSSM* suggest that “standard algorithms” are one means to achieve this fluency. An algorithm (named for the ninth-century Arabian mathematician Abu al Khwarizmi) is a systematic procedure used to accomplish an operation. In the 1998 National Council of Teachers of Mathematics (NCTM) Yearbook, *Teaching and Learning Algorithms in School Mathematics*, Usiskin stated that “algorithms are generalizations that embody one of the main reasons for studying mathematics—to find ways of solving classes of problems. When we know an algorithm, we complete not just one task but all tasks of a particular kind and we are guaranteed an answer or answers. The power of an algorithm derives from the breadth of its applicability.” (p. 10)
This section focuses on developing and understanding algorithms involving addition and subtraction. We develop alternative algorithms as well as standard algorithms.

### Addition Algorithms

In teaching mathematics to young children, it is important that we support them in the transition from concrete to abstract thinking by using techniques that parallel their developmental processes. To help children understand the use of paper-and-pencil algorithms, they should explore addition by first using manipulatives. If children can touch and move around items such as chips, bean sticks, and an abacus or use base-ten blocks, they can be led (and often will proceed naturally on their own) to the creation of algorithms for addition. In what follows, we use base-ten blocks to illustrate the development of an algorithm for whole-number addition.

Suppose we want to add $14 + 23$. We start with a concrete model in Figure 3-12(a), move to the expanded algorithm in Figure 3-12(b), and then to the standard algorithm in Figure 3-12(c).

![Concrete model](image1)

![Expanded algorithm](image2)

![Standard algorithm](image3)

Figure 3-12

A more formal justification for this addition not usually presented at the elementary level is the following:

\[
14 + 23 = (1 \cdot 10 + 4) + (2 \cdot 10 + 3) \quad \text{Place value}
\]

\[
= (1 \cdot 10 + 2 \cdot 10) + (4 + 3) \quad \text{Commutative and associative properties of addition}
\]

\[
= (1 + 2)10 + (4 + 3) \quad \text{Distributive property of multiplication over addition}
\]

\[
= 3 \cdot 10 + 7 \quad \text{Single-digit addition facts}
\]

\[
= 37 \quad \text{Place value}
\]

On the student page (page 129), we see an example of adding two-digit numbers with regrouping using base-ten blocks. Kim’s way leads to the expanded algorithm and Henry’s way leads to the standard algorithm. Each of these algorithms is discussed in more detail on page 130. Notice that on the student page, students are asked to estimate their answers before performing an algorithm. This is good practice and leads to the development of number sense and also makes students consider if their answers are reasonable. Study the student page and answer the Talk About It questions.
Lesson 3-1

Key Idea
You can break apart numbers, using place value, to add.

Vocabulary
* regroup

Adding Two-Digit Numbers

How do you add two-digit numbers?

Example
Cal counted 46 ladybugs on a log and 78 more on some bushes. How many ladybugs did he count all together?

Find 46 + 78.

Estimate: 46 rounds to 50, 78 rounds to 80.

50 + 80 = 130, so the answer should be about 130.

What You Think | What You Write
---|---
**Kim’s Way**
- Add the ones. 
  \( 6 + 8 = 14 \) ones 
- Add the tens. 
  \( 4 \) tens + \( 7 \) tens = \( 11 \) tens = \( 110 \) 
- Find the sum. 
  \( 11 \) tens \( 14 \) ones

**Henry’s Way**
- Add the ones. 
  \( 6 + 8 = 14 \) ones 
- **Regroup** 14 ones into \( 1 \) ten \( 4 \) ones. 
- Add the tens. 
  \( 1 \) ten + \( 4 \) tens + \( 7 \) tens = \( 12 \) tens 
- Find the sum. 
  \( 14 \) ones = \( 1 \) ten \( 4 \) ones

Cal counted 124 ladybugs all together.

**Talk About It**

1. Why did Henry write a small 1 above the 4 in the tens place?
2. Why should you estimate when adding two-digit numbers?
Once children have mastered the use of concrete models with regrouping, they should be ready to use the expanded and standard algorithms. Figure 3-13 shows the computation $37 + 28$ using both algorithms. In Figure 3-13(b), you will notice that when there were more than 10 ones, we regrouped 10 ones as a ten and then added the tens. Notice that the words *regroup* or *trade* are now commonly used in the elementary classroom to describe what we used to call *carrying*.

\[
\begin{array}{c@{}c@{}c@{}c@{}c@{}c@{}c}
\ & 3 & 7 \\
+ & 2 & 8 \\
\hline
& 1 & 5 \\
\end{array}
\quad \text{(Add ones)}
\quad \begin{array}{c@{}c@{}c@{}c@{}c@{}c@{}c}
\ & 3 & 7 \\
+ & 2 & 8 \\
\hline
& 6 & 5 \\
\end{array}
\quad \text{(Add the ones, regroup, and add the tens)}
\]

\[
\begin{array}{c@{}c@{}c@{}c@{}c@{}c@{}c}
\ & + 5 & 0 \\
\hline
& + 6 & 5 \\
\end{array}
\quad \text{(Add tens)}
\]

\begin{align*}
\text{Expanded algorithm} & \quad \text{Standard algorithm} \\
\end{align*}

\text{Figure 3-13}

Next we add two three-digit numbers involving two regroupings. Figure 3-14 shows how to add $186 + 127$ using base-ten blocks and how this concrete model carries over to the standard algorithm.

\begin{align*}
\text{Concrete Model} & \quad \text{Standard Algorithm} \\
1. \text{ Add the ones and regroup.} & \quad 186 + 127 \\
\quad 6 \text{ ones} + 7 \text{ ones} = 13 \text{ ones} & \quad \frac{1}{3} \\
\quad 13 \text{ ones} = 1 \text{ ten} + 3 \text{ ones} & \\
2. \text{ Add the tens and regroup.} & 186 + 127 \\
\quad 1 \text{ ten} + 8 \text{ tens} + & \frac{11}{313} \\
\quad 2 \text{ tens} = 11 \text{ tens} & \\
\quad 11 \text{ tens} = 1 \text{ hundred} + 1 \text{ ten} & \\
3. \text{ Add the hundreds.} & 186 + 127 \\
\quad 1 \text{ hundred} + 1 \text{ hundred} + & \frac{11}{313} \\
\quad 1 \text{ hundred} = 3 \text{ hundreds} & \\
\quad 186 + 127 = 313 & \\
\end{align*}

\text{Figure 3-14}
Students often develop algorithms on their own, and learning can occur by investigating how and if various algorithms work. Addition of whole numbers using blocks has a natural carryover to the expanded form and trading used earlier. For example, consider the following addition:

\[
\begin{array}{c}
376 \\
459 \\
8716
\end{array}
\quad \begin{array}{c}
3 \cdot 10^2 + 7 \cdot 10 + 6 \\
4 \cdot 10^2 + 5 \cdot 10 + 9 \\
8 \cdot 10^3 + 7 \cdot 10^2 + 1 \cdot 10 + 6
\end{array}
\]

\[
\frac{3 \cdot 10^2 + 7 \cdot 10 + 6}{4 \cdot 10^2 + 5 \cdot 10 + 9} + \frac{8 \cdot 10^3 + 7 \cdot 10^2 + 1 \cdot 10 + 6}{8 \cdot 10^3 + 14 \cdot 10^2 + 13 \cdot 10 + 21}
\]

To complete the addition, trading is used. However, consider an analogous algebra problem of adding polynomials:

\[
(3x^2 + 7x + 6) + (4x^2 + 5x + 9) + (8x^3 + 7x^2 + x + 6)
\]

or

\[
3x^2 + 7x + 6 \\
4x^2 + 5x + 9 \\
8x^3 + 7x^2 + x + 6
\]

\[
\frac{3x^2 + 7x + 6}{4x^2 + 5x + 9} + \frac{8x^3 + 7x^2 + x + 6}{8x^3 + 14x^2 + 13x + 21}
\]

Note that if \(x = 10\), the addition is the same as given earlier. Also note that knowledge of place value in addition problems aids in algebraic thinking. Next we explore several algorithms that have been used throughout history.

**Left-to-Right Algorithm for Addition**

Since children learn to read from left to right, it might seem natural that they may try to add from left to right. When working with base-ten blocks, many children in fact do combine the larger pieces first and then move to combining the smaller pieces. This method has the advantage of emphasizing place value. A left-to-right algorithm is as follows:

\[
\begin{array}{c}
568 \\
757 \\
1200 \\
110 \\
15 \\
1325
\end{array}
\quad \begin{array}{c}
568 \\
+ 757 \\
1200 \\
+ 110 \\
+ 15 \\
1325
\end{array}
\]

\[
(500 + 700) \rightarrow 1200 \\
(60 + 50) \rightarrow 110 \\
(8 + 7) \rightarrow 15 \\
\]

Explain why this technique works and try it with 9076 + 4689.

**Lattice Algorithm for Addition**

We introduce this algorithm by working through an addition involving two four-digit numbers. For example,

\[
\begin{array}{c}
3 \text{ } 5 \text{ } 6 \text{ } 7 \\
+ 5 \text{ } 6 \text{ } 7 \text{ } 8
\end{array}
\]

\[
\begin{array}{c}
9 \text{ } 8 \text{ } 1 \text{ } 5 \text{ } 2 \text{ } 4 \text{ } 5
\end{array}
\]

To use this algorithm, add the single-digit numbers by place value on top to the single digit numbers on the bottom and record the results in a lattice. Then add the sums along the diagonals, as shown. This is very similar to the expanded algorithm introduced earlier. Try this technique with 4578 + 2691.
Scratch Algorithm for Addition

The scratch algorithm for addition is often referred to as a *low-stress algorithm* because it allows students to perform complicated additions by doing a series of additions that involve only two single digits. An example follows:

1. \[
\begin{array}{c}
87 \\
62 \\
+ 49 \\
\hline
186
\end{array}
\]

Add the numbers in the units place starting at the top. When the sum is 10 or more, record this sum by scratching a line through the last digit added and writing the number of units next to the scratched digit. For example, since \(7 + 5 = 12\), the “scratch” represents 10 and the 2 represents the units.

2. \[
\begin{array}{c}
87 \\
62 \\
+ 49 \\
\hline
186
\end{array}
\]

Continue adding the units, including any new digits written down. When the addition again results in a sum of 10 or more, as with \(2 + 9 = 11\), repeat the process described in (1).

3. \[
\begin{array}{c}
287 \\
62 \\
+ 49 \\
\hline
358
\end{array}
\]

When the first column of additions is completed, write the number of units, 1, below the addition line in the proper place value position. Count the number of scratches, 2, and add this number to the second column.

4. \[
\begin{array}{c}
2807 \\
62 \\
+ 49 \\
\hline
3506
\end{array}
\]

Repeat the procedure for each successive column until the last column with non-zero values. At this stage, sum the scratches and place the number to the left of the current value.

Try this technique with \(56 + 23 + 34 + 67\).

Subtraction Algorithms

As with addition, we can use base-ten blocks to provide a concrete model for subtraction. Consider how the base-ten blocks are used to perform the subtraction \(243 - 61\): First we represent 243 with 2 flats, 4 longs, and 3 units, as shown in Figure 3-15.

To subtract 61 from 243, we try to remove 6 longs and 1 unit from the blocks in Figure 3-15. We can remove 1 unit, as in Figure 3-16.

To remove 6 longs from Figure 3-16, we have to trade 1 flat for 10 longs, as shown in Figure 3-17.
Student work with base-ten blocks along with discussions and recorded work lead to the development of the standard algorithm as seen on the student page on page 134. Work through the student page (a)–(f).

**NOW TRY THIS 3-8** Use base-ten blocks and addition to check that $243 - 61 = 182$.

---

### Equal-Additions Algorithm

The equal-additions algorithm for subtraction is based on the fact that the difference between two numbers does not change if we add the same amount to both numbers. For example, $93 - 27 = (93 + 3) - (27 + 3)$. Thus, the difference can be computed as $96 - 30 = 66$. Using this approach, the subtraction on the student page could be performed as follows:

\[
\begin{align*}
255 & \rightarrow 255 + 7 \\
-163 & \rightarrow -(163 + 7) \\
& \rightarrow -170 \\
& \rightarrow -(170 + 30) \\
& \rightarrow -200 \\
& \rightarrow 292
\end{align*}
\]

Subtraction of whole numbers using blocks has a natural carryover to the expanded form and trading. For example, consider the following subtraction problem done earlier with blocks:

\[
\begin{align*}
243 - 61 & \rightarrow 2 \cdot 10^2 + 4 \cdot 10 + 3 \\
& - (6 \cdot 10 + 1) \\
& \rightarrow 1 \cdot 10^2 + 14 \cdot 10 + 3 \\
& - (6 \cdot 10 + 1) \\
& \rightarrow 1 \cdot 10^2 + (14 - 6) 10 + (3 - 1)
\end{align*}
\]

which results in 182. Note that to complete the subtraction, trading was used. As with addition, we see that understanding place value aids in subtraction computations.
Models for Subtracting Three-Digit Numbers

**How can you subtract with place-value blocks?**

Find $255 - 163$.

<table>
<thead>
<tr>
<th>What You Show</th>
<th>What You Write</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Show 255 with place-value blocks.</td>
<td>[Diagram of blocks showing 255]</td>
</tr>
<tr>
<td>b. Subtract the ones. Regroup if needed.</td>
<td>[Diagram of blocks showing subtraction]</td>
</tr>
<tr>
<td>5 &gt; 3. No regrouping is needed.</td>
<td>[Result: 192]</td>
</tr>
<tr>
<td>c. Subtract the tens. Regroup if needed.</td>
<td>[Diagram of blocks showing subtraction]</td>
</tr>
<tr>
<td>5 tens &lt; 6 tens. So, regroup 1 hundred for 10 tens.</td>
<td>[Result: 192]</td>
</tr>
<tr>
<td>d. Subtract the hundreds.</td>
<td>[Diagram of blocks showing subtraction]</td>
</tr>
<tr>
<td>e. Find the value of the remaining blocks in Step d: 9 tens 2 ones = 92, so $255 - 163 = 92$.</td>
<td></td>
</tr>
<tr>
<td>f. In Step b, did you have to regroup to subtract the ones? Explain.</td>
<td></td>
</tr>
<tr>
<td>g. In Step c, did you have to regroup to subtract the tens? Explain.</td>
<td></td>
</tr>
<tr>
<td>h. Use place-value blocks to subtract.</td>
<td>$243 - 72$, $145 - 126$, $223 - 156$</td>
</tr>
</tbody>
</table>
NOW TRY THIS 3-9 Jessica claims that a method similar to equal additions for subtraction also works for addition. She says that in an addition problem, “you may add the same amount to one number as you subtract from the other.” For example, \(68 + 29 = (68 - 1) + (29 + 1)\). Thus, the sum can be computed as \(67 + 30 = 97\) or as \((68 + 2) + (29 - 2) = 70 + 27 = 97\). (i) Explain why this method is valid and (ii) use it to compute \(97 + 69\).

Understanding Addition and Subtraction in Bases Other Than Ten

A look at computation in other bases may provide insight into computation in base ten. Use of multibase blocks may be helpful in building an addition table for different bases and is highly recommended. Table 3-1 is a base-five addition table.

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
</tbody>
</table>

NOW TRY THIS 3-10 Write each of the following as numerals in base five:

a. \(444_{\text{five}} + 1_{\text{five}}\)
b. \(13_{\text{five}} + 13_{\text{five}}\)

Using the addition facts in Table 3-1, we can develop algorithms for base-five addition similar to those we used for base-ten addition. We show the computation using a concrete model in Figure 3-19(a); in Figure 3-19(b), we use an expanded algorithm; in Figure 3-19(c), we use the standard algorithm.
The subtraction facts for base five can also be derived from the addition-facts table by using the definition of subtraction. For example, to find $12_{\text{five}} - 4_{\text{five}}$, recall that $12_{\text{five}} - 4_{\text{five}} = c_{\text{five}}$ if, and only if, $c_{\text{five}} + 4_{\text{five}} = 12_{\text{five}}$. From Table 3-1, we see that $c = 3_{\text{five}}$. An example of subtraction involving regrouping, $32_{\text{five}} - 14_{\text{five}}$, is developed in Figure 3-20.

![Figure 3-20](image)

**NOW TRY THIS 3-11**

a. Build an addition table for base two.

b. Use the addition table from part (a) to perform (i) $1111_{\text{two}} + 111_{\text{two}}$ (ii) $1101_{\text{two}} - 111_{\text{two}}$.

**BRAIN TEASER** The number on a license plate consists of five digits. When the license plate is looked at upside down, you can still read it, but the value of the upside-down number is 78,633 greater than the real license number. What is the license number?

---

**Assessment 3-2A**

1. Find the missing digits in each of the following:
   a. \[ \_ \_ \_ 1 \]
   b. \[ \_ 0 2 \_ 5 \]
   c. \[ \_ 4 0 2 \]
   d. \[ \_ 1 1 \_ 6 \]
   e. \[ \_ 3 1 4 8 \]
   f. \[ \_ 6 6 \_ \]

2. Make an appropriate drawing like the one in Figure 3-14 to show the use of base-ten blocks to compute $29 + 37$.

3. Place the digits 7, 6, 8, 3, 5, and 2 in the boxes to obtain
   a. the greatest sum.
   b. the least sum.

4. Tom’s diet allows only 1500 calories per day. For breakfast, Tom had skim milk (90 calories), a waffle with no syrup (120 calories), and a banana (119 calories). For lunch, he had $\frac{1}{4}$ cup of salad (185 calories) with mayonnaise (110 calories) and tea (0 calories). Then he had pecan pie (570 calories). Can he have dinner consisting of fish (250 calories), a $\frac{1}{4}$ cup of salad with no mayonnaise, and tea?

5. Wally kept track of last week’s money transactions. His salary was $150 plus $54 in overtime and $260 in tips. His transportation expenses were $22, his food expenses were $60, his laundry costs were $15, his entertainment expenditures were $58, and his rent was $185.
After expenses, did he have any money left? If so, how much?

6. In the following problem, the sum is correct but the order of the digits in each addend has been scrambled. Correct the addends to obtain the correct sum.

\[
\begin{array}{c}
283 + \\
+ 6315 \\
\hline
9059
\end{array}
\]

7. Use the equal-additions approach to compute each of the following:
   a. 93
      \[
      - 37
      \]
   b. 321
      \[
      - 38
      \]

8. Janet worked her addition problems by placing the partial sums as shown here:

\[
\begin{array}{c}
\phantom{569} + 645 \\
\hline
14 \\
10 \\
11 \\
\hline
1214
\end{array}
\]

   a. Use this method to work the following:
      (i) 687
      \[
      + 549 \\
      \hline
      \]
      (ii) 359
      \[
      + 673 \\
      \hline
      \]
   b. Explain why this algorithm works.

9. Analyze the following computations. Explain what is wrong in each case.
   a. 28
      \[
      + 75 \\
      \hline
      913
      \]
   b. 28
      \[
      + 75 \\
      \hline
      121
      \]
   c. 305
      \[
      - 259 \\
      \hline
      154
      \]
   d. 310
      \[
      - 259 \\
      \hline
      56
      \]

10. Give reasons for each of the following steps:
   \[
   16 + 31 = (1 \cdot 10 + 6) + (3 \cdot 10 + 1)
   = (1 + 3)10 + (6 + 1)
   = 4 \cdot 10 + 7
   = 47
   \]

11. In each of the following, justify the standard addition algorithm using place value of the numbers, the commutative and associative properties of addition, and the distributive property of multiplication over addition:
   a. 68 + 23
   b. 174 + 285
   c. 2458 + 793

12. Use the lattice algorithm to perform each of the following:
   a. 4358 + 3864
   b. 4923 + 9897

13. Perform each of the following operations using the bases shown:
   a. 43_{five} + 23_{five}
   b. 43_{five} - 23_{five}
   c. 432_{five} + 23_{five}
   d. 42_{five} - 23_{five}
   e. 110_{two} + 11_{two}
   f. 11001_{two} - 111_{two}

14. Construct an addition table for base eight.

15. Perform each of the following operations:
   a. 3 hr 36 min 58 sec
   b. 5 hr 36 min 38 sec
   + 3 hr 56 min 58 sec

16. Andrew's calculator was not functioning properly. When he pressed \(8 + 6 =\), the numeral 20 appeared on the display. When he pressed \(5 + 4 =\), 13 was displayed. When he pressed \(1 \times 3 =\), 9 was displayed. What do you think Andrew's calculator was doing?

17. Use scratch addition to perform the following:
   a. 432
      \[
      976 \\
      \hline
      1418
      \]
   b. 32_{five}
      \[
      13_{five} \\
      22_{five} \\
      43_{five} \\
      \hline
      12_{five}
      \]

18. Perform each of the following operations:
   a. 4 gross 4 doz 6 ones
      \[
      - 5 doz 9 ones
      \]
   b. 2 gross 9 doz 7 ones
      \[
      + 3 gross 5 doz 9 ones
      \]

19. Determine what is wrong with the following:
   \[
   \begin{array}{c}
   22_{five} \\
   + 33_{five} \\
   \hline
   55_{five}
   \end{array}
   \]

20. Fill in the missing numbers in each of the following:
   a. 2_{five}
      \[
      \]
      \[
      - 2 \quad 2_{five} \\
      \]
      \[
      - 0 \quad 3_{five}
      \]
21. Find the numeral to put in the blank to make each equation true. Do not convert to base ten.
   a. \(3423_{\text{five}} - \_ = 2132_{\text{five}}\)
   b. \(1101_{\text{two}} + \_ = 100000_{\text{two}}\)
   c. \(\text{TEE}_{\text{twelve}} - \_ = 1\)
   d. \(1000_{\text{five}} + \_ = 10000_{\text{five}}\)

22. A palindrome is any number that reads the same backward as forward, for example, 121 and 2332. Try the following.
   a. Begin with any multi-digit number. Is it a palindrome? If not, reverse the digits and add this reversed number to the original number. Is the result a palindrome? If not, repeat the procedure until a palindrome is obtained. For example, start with 78. Because 78 is not a palindrome, we add:

\[78 + 87 = 165\]

Because 165 is not a palindrome, we add:

\[165 + 561 = 726\]

Again, 726 is not a palindrome, so we add:

\[726 + 627 = 1353\]

Finally, 1353 + 3531 yields 4884, which is a palindrome.

b. Try this method with the following numbers:
   (i) 93
   (ii) 588
   (iii) 2003

b. Find a number for which the procedure described takes more than five steps to form a palindrome.

---

**Assessment 3-2B**

1. Find the missing digits in each of the following:
   a. \(3 - 1 5 9\)
     
   b. \(1 - 8 3 0 9\)
     
2. Make an appropriate drawing like the one in Figure 3-12 to show the use of base-ten blocks to compute 46 + 38.

3. Place the digits 7, 6, 8, 3, 5, and 2 in the boxes to obtain
   a. the greatest difference.
   b. the least difference.

4. In the following problem, the sum is correct but the order of the digits in each addend has been scrambled. Correct the addends to obtain the correct sum.

\[
8354 + 3456 = 11810
\]

5. Use the equal-additions approach to compute each of the following:
   a. 86
   b. 582

6. Janet worked her addition problems by placing the partial sums as shown here:

\[
\begin{align*}
  569 & \quad + \quad 645 \\
  14 \quad & \quad 10 \\
  11 \quad & \quad 1214 \\
\end{align*}
\]

---

7. Analyze the following computations. Explain what is wrong in each case.
   a. 135
     
   b. 87
     
   c. 57
     
   d. 56
     
8. George is cooking an elaborate meal for Thanksgiving. He can cook only one thing at a time in his microwave oven. His turkey takes 75 min; the pumpkin pie takes 18 min; rolls take 45 sec; and a cup of coffee takes 30 sec to heat. How much time does he need to cook the meal?

9. Give reasons for each of the following steps:

\[
123 + 45 = (1 \cdot 10^2 + 2 \cdot 10 + 3) + (4 \cdot 10 + 5)
\]

10. In each of the following justify the standard addition algorithm using place value of the numbers, the
Section 3-2 Algorithms for Whole-Number Addition and Subtraction

commutative and associative properties of addition, and the distributive property of multiplication over addition:

a. $46 + 32$
b. $3214 + 783$

11. Use the lattice algorithm to perform each of the following:

a. $2345 + 8888$
b. $8713 + 4214$

12. Perform each of the following operations using the bases shown:

a. $43_{\text{five}} - 24_{\text{five}}$
b. $143_{\text{five}} + 23_{\text{five}}$
c. $32_{\text{five}} - 23_{\text{five}}$
d. $232_{\text{five}} + 43_{\text{five}}$
e. $110_{\text{two}} + 111_{\text{two}}$
f. $10001_{\text{two}} - 101_{\text{two}}$

13. Construct an addition table for base six.

14. Perform each of the following operations (2 c = 1 pt, 2 pt = 1 qt, 4 qt = 1 gal):

a. $1 \text{qt} 1 \text{pt} 1 \text{c}$

$+ 1 \text{pt} 1 \text{c}$

b. $1 \text{qt} 1 \text{c}$

$- 1 \text{pt} 1 \text{c}$

c. $1 \text{gal} 3 \text{qt} 1 \text{c}$

$- 4 \text{qt} 2 \text{c}$

15. The following is a supermagic square taken from an engraving called *Melancholia* by Dürer. Notice 1514 in the bottom row, the year the engraving was made.

<table>
<thead>
<tr>
<th>16</th>
<th>3</th>
<th>2</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>14</td>
<td>1</td>
</tr>
</tbody>
</table>

a. Find the sum of each row, the sum of each column, and the sum of each diagonal.
b. Find the sum of the four numbers in the center.
c. Find the sum of the four numbers in each corner.
d. Add 11 to each number in the square. Is the square still a magic square? Explain your answer.
e. Subtract 11 from each number in the square. Is the square still a magic square?

16. Use scratch addition to perform the following:

a. $537$

$+ 2345$

b. $41_{\text{six}}$

$+ 32_{\text{six}}$

$22_{\text{six}}$

$43_{\text{six}}$

$22_{\text{six}}$

$+ 54_{\text{six}}$

17. Determine what is wrong with the following:

$23_{\text{six}}$

$+ 43_{\text{six}}$

$66_{\text{six}}$

18. Find the numeral to put in the blank to make each equation true. Do not convert to base ten.

a. $342_{\text{five}} - \_ = 213_{\text{five}}$
b. $1101_{\text{two}} - \_ = 1011_{\text{two}}$
c. $E08_{\text{twelve}} - \_ = 9_{\text{twelve}}$
d. $100_{\text{two}} + \_ = 10000_{\text{two}}$

19. The Hawks played the Elks in a basketball game. Based on the following information, complete the scoreboard showing the number of points scored by each team during each quarter and the final score of the game.

<table>
<thead>
<tr>
<th>Teams</th>
<th>Quarters</th>
<th>Final Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawks</td>
<td>1 2 3 4</td>
<td></td>
</tr>
<tr>
<td>Elks</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. The Hawks scored 15 points in the first quarter.
b. The Hawks were behind by 5 points at the end of the first quarter.
c. The Elks scored 5 more points in the second quarter than they did in the first quarter.
d. The Hawks scored 7 more points than the Elks in the second quarter.
e. The Elks outscored the Hawks by 6 points in the fourth quarter.
f. The Hawks scored a total of 120 points in the game.
g. The Hawks scored twice as many points in the third quarter as the Elks did in the first quarter.
h. The Elks scored as many points in the third quarter as the Hawks did in the first two quarters combined.

20. a. Place the numbers 24 through 32 in the following circles so that the sums are the same in each direction:

b. How many different numbers can be placed in the middle to obtain a solution?
Mathematical Connections 3-2

Communication

1. Discuss the merit of the following algorithm for addition where we first add the ones, then the tens, then the hundreds, and then the total:

\[
\begin{array}{c}
479 \\
+ 385 \\
\hline
14 \\
150 \\
+ 700 \\
\hline
864
\end{array}
\]

2. The following example uses a regrouping approach to subtraction. Discuss the merit of this approach in teaching subtraction.

\[
\begin{array}{c}
843 \\
- 568 \\
\hline
= 275
\end{array}
\]

3. Tira, a fourth grader, performs addition by adding and subtracting the same number. She added as follows:

\[
\begin{array}{c}
39 \\
+ 84 \\
\hline
= 123
\end{array}
\]

How would you respond if you were her teacher?

4. Explain why the scratch addition algorithm works.

5. The equal-additions algorithm was introduced in this section. The following shows how this algorithm works for 1464 - 687:

\[
\begin{array}{c}
1 4 6 1 4 \\
\underline{- 6 0 8 7 7} \\
\hline
1 7 8 7 7 \\
(Add 1 ten to the 4 ones to get 14 ones.) \\
(Add 1 ten to the 8 tens to get 19 tens.) \\
(Subtract the ones.)
\end{array}
\]

Now we move to the next column:

\[
\begin{array}{c}
1 4 6 1 4 \\
\underline{- 7 0 8 7 7} \\
\hline
7 9 7 7 \\
(Add 1 hundred to the 6 hundreds to get 7 hundreds.) \\
(Subtract the 9 tens from the 16 tens and then the 7 hundreds from the 14 hundreds.)
\end{array}
\]

6. Cathy found her own algorithm for subtraction. She subtracted as follows:

\[
\begin{array}{c}
97 \\
- 28 \\
\hline
7 \\
+ 70 \\
\hline
69
\end{array}
\]

How would you respond if you were her teacher?

7. Discuss why the words regroup and trade are used rather than carry and borrow for whole-number addition and subtraction algorithms.

8. Consider the following subtraction algorithm.

a. Explain how it works.

b. Use this algorithm to find 787 - 398.

\[
\begin{array}{c}
585 \\
\underline{- 277} \\
\hline
308
\end{array}
\]

Open-Ended

9. Search for or develop an algorithm for whole-number addition or subtraction and write a description of your algorithm so that others can understand and use it.

Cooperative Learning

10. In this section you have been exposed to many different algorithms. Discuss in your group whether children should be encouraged to develop and use their own algorithms for whole-number addition and subtraction or whether they should be taught only one algorithm per operation and all students should use only one algorithm.

Questions from the Classroom

11. To find 68 - 19, Joe began by finding 9 - 8. How do you help?

12. Jill subtracted 415 - 212 and got 303. She asked if this was correct. How would you respond?

13. Betsy found 518 - 49 = 469. She was not sure she was correct so she tried to check her answer by adding 518 + 49. How could you help her?
14. A child is asked to compute $7 + 2 + 3 + 8 + 11$ and writes $7 + 2 = 9 + 3 = 12 + 8 = 20 + 11 = 31$. Noticing that the answer is correct, if you were the teacher how would you react?

Review Problems
15. Is the set $\{1, 2, 3\}$ closed under addition? Why?
16. Give an example of the associative property of addition of whole numbers.

National Assessment of Educational Progress (NAEP) Questions

The figure above represents 237. Which number is more than 237?

a. 244  b. 249  c. 251  d. 377

NAEP 2007, Grade 4

The Ben Franklin Bridge was 75 years old in 2001. In what year was the bridge 50 years old?


NAEP 2007, Grade 4

LABORATORY ACTIVITY

1. One type of Japanese abacus, soroban, is shown in Figure 3-21(a). In this abacus, a bar separates two sets of bead counters. Each counter above the bar represents 5 times the counter below the bar. Numbers are illustrated by moving the counter toward the bar. The number 7632 is pictured. Practice demonstrating and adding numbers on this abacus.

2. The Chinese abacus, suan pan (see Figure 3-21(b)), is still in use today. This abacus is similar to the Japanese abacus but has two counters above the bar and 5 counters below the bar. The number 7632 is also pictured on it. Practice demonstrating and adding numbers on this abacus. Compare the ease of using the two versions.
Multiplication and Division of Whole Numbers

In the grade 3 Focal Points we find the following concerning multiplication and division of whole numbers:

Students understand the meanings of multiplication and division of whole numbers through the use of representations (e.g., equal-sized groups, arrays, area models, and equal “jumps” on number lines for multiplication, and successive subtraction, partitioning, and sharing for division). They use properties of addition and multiplication (e.g., commutativity, associativity, and the distributive property) to multiply whole numbers and apply increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving basic facts. By comparing a variety of solution strategies, students relate multiplication and division as inverse operations. (p. 15)

Further in the grade 3 Focal Points, we see the connection between studying multiplication and division of whole numbers and the study of algebra.

Understanding properties of multiplication and the relationship between multiplication and division is a part of algebra readiness that develops at grade 3. The creation and analysis of patterns and relationships involving multiplication and division should occur at this grade level. Students build a foundation for later understanding of functional relationships by describing relationships in context with such statements as, “The number of legs is 4 times the number of chairs.” (p. 15)

These quotes from the Focal Points set the tone and agenda for this section. We discuss representations that can be used to help students understand the meanings of multiplication and division. We develop the distributive property of multiplication over addition along with the relationship of multiplication and division as inverse operations.

Multiplication of Whole Numbers

In this section, we explore the kind of problems that Grampa is having in the Peanuts cartoon. Why do you think he would have more troubles with “9 times 8” rather than “3 times 4”? If multiplication facts are only memorized, they may be forgotten. If students have a conceptual understanding of the basic facts, then all of the basic facts can be determined even if not automatically recalled.
Repeated-Addition Model

On the student page on page 144 we see that if we have 4 groups of 3 brushes, we can use addition to put the groups together. When we put equal-sized groups together we can use multiplication. We can think of this as combining 4 sets of 3 objects into a single set. The 4 sets of 3 suggest the following addition:

\[3 + 3 + 3 + 3 = 12\]

We write \(3 + 3 + 3 + 3\) as \(4 \cdot 3\) and say “four times three” or “three multiplied by four.” The advantage of the multiplication notation over repeated addition is evident when the number of addends is great, for example, if we have 25 groups of 3 brushes, we could find the total number of brushes by adding 25 threes or \(25 \cdot 3\).

The repeated-addition model can be illustrated in several ways, including number lines and arrays. For example, using colored rods of length 4, we could show that the combined length of five of the rods can be found by joining the rods end-to-end, as in Figure 3-22(a). Figure 3-22(b) shows the process using arrows on a number line.

The constant feature on a calculator can help relate multiplication to addition. Students can find products on the calculator without using the key. For example, if a calculator has the constant feature, then \(5 \times 3\) can be found by starting with and pressing \(5 \times 3 = \). Each press of the equal sign will add 3 to the display. (Some calculators will work differently.)

As pointed out in the Research Note, access to only the “repeated-addition” model for multiplication can lead to misunderstanding. In this section, we introduce three other multiplication models: the array and area models and the Cartesian-product model.

Research Note

Students learning multiplication as a conceptual operation need exposure to a variety of models (for example, array and area). Access only to the “multiplication as repeated addition” and the term times leads to basic misunderstandings of multiplication that complicate future extensions to decimals and fractions (Bell et al. 1989; English and Halford 1995).

Historical Note

William Oughtred (1574–1660), an English mathematician, was interested in mathematical symbols. He was the first to introduce the “St. Andrew’s cross” \((\times)\) as the symbol for multiplication. This symbol was not readily adopted because, as Gottfried Wilhelm von Leibniz (1646–1716) objected, it was too easily confused with the letter \(x\). Leibniz used the dot \((\cdot)\) for multiplication, which has become common.
Multiplication as Repeated Addition

**Activity**

**How can you find the total?**

There are 4 groups of 3 paintbrushes.

You can use addition to put together groups.

\[ 3 + 3 + 3 + 3 = 12 \quad \text{Addition sentence} \]

When you put together equal groups, you can also use **multiplication**.

What You Say: 4 times 3 equals 12

What You Write: \[ 4 \times 3 = 12 \quad \text{Multiplication sentence} \]

**a.** Write an addition sentence and a multiplication sentence to show the total number of counters below.

\[ \star \star \star \star \star \]

**b.** Use counters and draw a picture to show the groups described below. For each picture, write an addition sentence and a multiplication sentence to show how many counters in all.

- 5 groups of 2
- 4 groups of 5
- 3 groups of 3

Source: Scott Foresman-Addison Wesley Math, Grade 3, 2008 (p. 260).
The Array and Area Models

Another representation useful in exploring multiplication of whole numbers is an array. An array is suggested when we have objects in equal-sized rows, as in Figure 3-23.

In Figure 3-24(a), we cross sticks to create intersection points, thus forming an array of points. The number of intersection points on a single vertical stick is 4 and there are 5 sticks, forming a total of $5 \cdot 4$ points in the array. In Figure 3-24(b), the area model is shown as a 4-by-5 grid. The number of unit squares required to fill in the grid is 20. These models motivate the following definition of multiplication of whole numbers.

![Figure 3-23](image)

**Figure 3-23**

**Definition of Multiplication of Whole Numbers**

For any whole numbers $a$ and $n \neq 0$,

$$n \cdot a = a + a + a + \ldots + a.$$  \(n\) terms

If $n = 0$, then $0 \cdot a = 0$.

**REMARK** We typically write $n \cdot a$ as $na$, where $a$ is not a number but a variable.

Cartesian-Product Model

The Cartesian-product model offers another way to discuss multiplication. Suppose you can order a soyburger on light or dark bread with one condiment: mustard, mayonnaise, or horseradish. To show the number of different soyburger orders that a waiter could write for...
the cook, we use a *tree diagram*. The ways of writing the order are listed in Figure 3-25, where the bread is chosen from the set \( B = \{ \text{light, dark} \} \) and the condiment is chosen from the set \( C = \{ \text{mustard, mayonnaise, horseradish} \} \).

Each order can be written as an ordered pair, for example, (dark, mustard). The set of ordered pairs forms the Cartesian product \( B \times C \). The Fundamental Counting Principle tells us that the number of ordered pairs in \( B \times C \) is \( 2 \cdot 3 \).

The preceding discussion demonstrates how multiplication of whole numbers can be defined in terms of Cartesian products. Thus, an alternative definition of multiplication of whole numbers is as follows:

### Alternative Definition of Multiplication of Whole Numbers

For finite sets \( A \) and \( B \), if \( n(A) = a \) and \( n(B) = b \), then \( a \cdot b = n(A \times B) \).

In this alternative definition, sets \( A \) and \( B \) do not have to be disjoint. The expression \( a \cdot b \), or simply \( ab \), is the **product** of \( a \) and \( b \), and \( a \) and \( b \) are **factors**. Note that \( A \times B \) indicates the Cartesian product, not multiplication. We multiply numbers, not sets.

### NOW TRY THIS 3-12

How would you use the repeated-addition definition of multiplication to explain to a child unfamiliar with the Fundamental Counting Principle that the number of possible outfits consisting of a shirt and pants combination—given 6 shirts and 5 pairs of pants—is \( 6 \cdot 5 \)?

The following problems illustrate each of the models shown for multiplication. In all five problems, the answer can be thought of using a different model. Work through each problem using the suggested model.

1. **Repeated-addition model.** One piece of gum costs 5¢; how much do three pieces cost?
2. **Number-line model.** If Al walks 5 mph for 3 hr, how far has he walked?
3. **Array model.** A panel of stamps has 4 rows of 5 stamps. How many stamps are there in a panel?
4. **Area model.** If a carpet is 5 ft by 3 ft, what is the area of the carpet?
5. **Cartesian-product model.** Al has 5 shirts and 3 pairs of pants; how many different shirt-pants combinations are possible?
Properties of Whole-Number Multiplication

The set of whole numbers is closed under multiplication. That is, if we multiply any two whole numbers, the result is a unique whole number. This property is referred to as the closure property of multiplication of whole numbers. Multiplication on the set of whole numbers, like addition, has the commutative, associative, and identity properties.

The associative property of multiplication of whole numbers can be illustrated as follows. Suppose and In Figure 3-27(a), we see a picture of blocks. In Figure 3-27(b), we see the same blocks, this time arranged as By the commutative property this can be written as Because both sets of blocks in Figure 3-27(a) and (b) compress to the set shown in Figure 3-27(c), we see that The associative property is useful in computations such as the following:

\[
3 \cdot 40 = 3(4 \cdot 10) = (3 \cdot 4)10 = 12 \cdot 10 = 120
\]

Figure 3-27

The commutative property of multiplication of whole numbers is illustrated easily by building a 3-by-5 grid and then turning it sideways, as shown in Figure 3-26. We see that the number of 1 x 1 squares present in either case is 15; that is, \(3 \cdot 5 = 5 \cdot 3\). The commutative property can be verified by recalling that \(n(A \times B) = n(B \times A)\).

The associative property of multiplication of whole numbers can be illustrated as follows. Suppose \(a = 3, b = 5,\) and \(c = 4\). In Figure 3-27(a), we see a picture of 3(5 · 4) blocks. In Figure 3-27(b), we see the same blocks, this time arranged as 4(3 · 5). By the commutative property this can be written as \((3 \cdot 5)4\). Because both sets of blocks in Figure 3-27(a) and (b) compress to the set shown in Figure 3-27(c), we see that \(3(5 \cdot 4) = (3 \cdot 5)4\). The associative property is useful in computations such as the following:

\[
3 \cdot 40 = 3(4 \cdot 10) = (3 \cdot 4)10 = 12 \cdot 10 = 120
\]

The multiplicative identity for whole numbers is 1. For example, \(3 \cdot 1 = 1 + 1 + 1 = 3\). In general, for any whole number \(a\),

\[
a \cdot 1 = 1 + 1 + 1 + \ldots + 1 = a
\]

\(a\) terms

Thus, \(a \cdot 1 = a\), which, along with the commutative property for multiplication implies that \(a \cdot 1 = a = 1 \cdot a\). Cartesian products can also be used to show that \(a \cdot 1 = a = 1 \cdot a\).
Next, consider multiplication involving 0. For example, $0 \cdot 6$ by definition means we have 0 6’s or 0. Also $6 \cdot 0 = 0 + 0 + 0 + 0 + 0 + 0 = 0$. Thus we see that multiplying 0 by 6 or 6 by 0 yields a product of 0. This is an example of the zero multiplication property. This property can also be verified by using the definition of multiplication in terms of Cartesian products. In algebra, $3x$ means $3 \times x$ or $x + x + x$. Therefore, $0 \cdot x$ means 0 sets of $x$, or 0.

**The Distributive Property of Multiplication over Addition and Subtraction**

We now investigate the basis for understanding multiplication algorithms for whole numbers. The area of the large rectangle in Figure 3-28 equals the sum of the areas of the two smaller rectangles and hence $5(3 + 4) = 5 \cdot 3 + 5 \cdot 4$.

![Figure 3-28](image)

The properties of addition and multiplication of whole numbers also can be used to justify this result:

$$5(3 + 4) = (3 + 4) + (3 + 4) + (3 + 4) + (3 + 4) + (3 + 4)$$

**Definition of multiplication**

$$= (3 + 3 + 3 + 3 + 3) + (4 + 4 + 4 + 4 + 4)$$

**Commutative and associative properties of addition**

$$= 5 \cdot 3 + 5 \cdot 4 \quad \text{Definition of multiplication}$$

Note that $5(3 + 4)$ can be thought of as $5 (3 + 4)'s$.

This example illustrates the distributive property of multiplication over addition for whole numbers. A similar property of subtraction is also true. Because in algebra it is customary to write $a \cdot b$ as $ab$, we state the distributive property of multiplication over addition and the distributive property of multiplication over subtraction as follows:

**Theorem 3–6: Distributive Property of Multiplication over Addition for Whole Numbers**

For any whole numbers $a$, $b$, and $c$,

$$a(b + c) = ab + ac$$

**Theorem 3–7: Distributive Property of Multiplication over Subtraction for Whole Numbers**

For any whole numbers $a$, $b$, and $c$ with $b > c$,

$$a(b - c) = ab - ac$$
The distributive property can be written as

\[ ab + ac = a(b + c) \]

This is commonly referred to as factoring. Thus, the factors of \( ab + ac \) are \( a \) and \( b + c \).

Students find the distributive property of multiplication over addition useful when doing mental mathematics. For example, \( 13 \cdot 7 = (10 + 3)7 = 10 \cdot 7 + 3 \cdot 7 = 70 + 21 = 91 \).

The distributive property of multiplication over addition is important in the study of algebra and in developing algorithms for arithmetic operations. For example, it is used to combine like terms when we work with variables, as in \( 3x + 5x = (3 + 5)x = 8x \) or \( 3ab + 2b = (3a + 2)b \).

**Example 3-2**

a. Use an area model to show that \( (x + y)(z + w) = xz + xw + yz + yw \).

b. Use the distributive property of multiplication over addition to justify the result in part (a).

**Solution**

a. Consider the rectangle in Figure 3-29, whose height is \( x + y \) and whose length is \( z + w \). The area of the entire rectangle is \( (x + y)(z + w) \). If we divide the rectangle into smaller rectangles as shown, we notice that the sum of the areas of the four smaller rectangles is \( xz + xw + yz + yw \). Because the area of the original rectangle equals the sum of the areas of the smaller rectangles, the result follows.

![Figure 3-29](image)

b. To apply the distributive property of multiplication over addition, we think about \( x + y \) as one number and proceed as follows:

\[
(x + y)(z + w) = (x + y)z + (x + y)w
\]

The distributive property of multiplication over addition

\[
= xz + yz + xw + yw
\]

The distributive property of multiplication over addition

\[
= xz + xw + yz + yw
\]

The commutative and associative properties of addition.
The properties of whole-number multiplication reduce the 100 basic multiplication facts involving numbers 0–9 that students have to learn. For example, 19 facts involve multiplication by 0, and 17 more have a factor of 1. Therefore, knowing the zero multiplication property and the identity multiplication property allows students to know 36 facts. Next, 8 facts are squares, such as $5 \cdot 5$, that students seem to know, and that leaves 56 facts. The commutative property cuts this number in half, because if students know $7 \cdot 9$, then they know $9 \cdot 7$. This leaves 28 facts that students can learn or use the associative and distributive properties to figure out. For example, $6 \cdot 5$ can be thought of as $(5 + 1)5 = 5 \cdot 5 + 1 \cdot 5$, or 30.

**Division of Whole Numbers**

We discuss division using three models: the *set (partition)* model, the *missing-factor* model, and the *repeated-subtraction* model.

**Set (Partition) Model**

Suppose we have 18 cookies and want to give an equal number of cookies to each of three friends: Bob, Dean, and Charlie. How many should each person receive? If we draw a picture, we can see that we can divide (or partition) the 18 cookies into three sets, with an equal number of cookies in each set. Figure 3-30 shows that each friend receives 6 cookies.

![Figure 3-30](image)

The answer may be symbolized as $18 \div 3 = 6$. Thus, $18 \div 3$ is the number of cookies in each of three disjoint sets whose union has 18 cookies. In this approach to division, we partition a set into a number of equivalent subsets.

**Missing-Factor Model**

Another strategy for dividing 18 cookies among three friends is to use the *missing-factor* model. If each friend receives $c$ cookies, then the three friends receive $3c$, or 18, cookies. Hence, $3c = 18$. Because $3 \cdot 6 = 18$, we have $c = 6$. We have answered the division computation by using multiplication. This leads us to the following definition of division of whole numbers:

**Definition of Division of Whole Numbers**

For any whole numbers $a$ and $b$, with $b \neq 0$, $a \div b = c$ if, and only if, $c$ is the unique whole number such that $b \cdot c = a$.

The number $a$ is the *dividend*, $b$ is the *divisor*, and $c$ is the *quotient*. Note that $a \div b$ can also be written as $\frac{a}{b}$ or $\overline{b}a$. 
Repeated-Subtraction Model

Suppose we have 18 cookies and want to package them in cookie boxes that hold 6 cookies each. How many boxes are needed? We could reason that if one box is filled, then we would have $18 - 6$ (or 12) cookies left. If one more box is filled, then there are $12 - 6$ (or 6) cookies left. Finally, we could place the last 6 cookies in a third box. This discussion can be summarized by writing $18 - 6 - 6 - 6 = 0$. We have found by repeated subtraction that $18 ÷ 6 = 3$. Treating division as repeated subtraction works well if there are no cookies left over. If there are cookies left over a nonzero remainder, will arise.

Calculators can illustrate the repeated subtraction operation. For example, consider $135 ÷ 15$. If the calculator has a constant key, $\boxed{\text{K}}$, press $135 - \boxed{\text{K}} 15 = \ldots$ and then count how many times you must press the $\boxed{\text{=}}$ key in order to make the display read 0. Calculators with a different constant feature may require a different sequence of entries. For example, on some calculators, we can press $135 \boxed{\text{=}} \boxed{\text{-}} 15 \boxed{\text{=}}$ and then count the number of times we press the $\boxed{\text{=}}$ key to make the display read $0$.

The Division Algorithm

Just as subtraction of whole numbers is not closed, division of whole numbers is not closed. For example, to find $27 ÷ 5$, we look for a whole number $c$ such that $5c = 27$.

Table 3-2 shows several products of whole numbers times 5. Since 27 is between 25 and 30, there is no whole number $c$ such that $5c = 27$. Because no whole number $c$ satisfies this equation, we see that $27 ÷ 5$ has no meaning in the set of whole numbers, and the set of whole numbers is not closed under division.

<table>
<thead>
<tr>
<th>$5 \cdot 1$</th>
<th>$5 \cdot 2$</th>
<th>$5 \cdot 3$</th>
<th>$5 \cdot 4$</th>
<th>$5 \cdot 5$</th>
<th>$5 \cdot 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
</tr>
</tbody>
</table>

Even though the set of whole numbers is not closed under division, practical applications with whole number divisions are common. For example, if 27 apples were to be divided among 5 students, each student would receive 5 apples and 2 apples would remain. The number 2 is the remainder. Thus, 27 contains five 5s with a remainder of 2. Observe that the remainder is a whole number less than 5. This operation is illustrated in Figure 3-31. The concept illustrated is the division algorithm.
When \( a \) is “divided” by \( b \) and the remainder is 0, we say that \( a \) is divisible by \( b \) or that \( b \) is a divisor of \( a \) or that \( b \) divides \( a \). By the division algorithm, \( a \) is divisible by \( b \) if \( a = bq \) for a unique whole number \( q \). Thus, 63 is divisible by 9 because \( 63 = 9 \times 7 \). Notice that 63 is also divisible by 7 and that the remainder is 0.

**Example 3-3**

If 123 is divided by a number and the remainder is 13, what are the possible divisors?

**Solution**

If 123 is divided by \( b \), then from the division algorithm we have

\[
123 = bq + 13 \quad \text{and} \quad b > 13
\]

Using the definition of subtraction, we have \( bq = 123 - 13 \), and hence \( 110 = bq \). Now we are looking for two numbers whose product is 110, where one number is greater than 13. Table 3-3 shows the pairs of whole numbers whose product is 110.

<table>
<thead>
<tr>
<th>1</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>55</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

We see that 110, 55, and 22 are the only possible values for \( b \) because each is greater than 13.

**NOW TRY THIS 3-13**

When the marching band was placed in rows of 5, one member was left over. When the members were placed in rows of 6, there was still one member left over. However, when they were placed in rows of 7, nobody was left over. What is the smallest number of members that could have been in the band?

**Relating Multiplication and Division as Inverse Operations**

In Section 3-1, we saw that subtraction and addition were related as inverse operations and we looked at fact families for both operations. In a similar way, division with remainder 0 and multiplication are related. Division is the inverse of multiplication. We can again see this by looking at fact families as shown on the grade 3 student page on page 153. Notice that question 1 in the *Talk About It* has students think of division using the repeated-subtraction model by skip counting backward from a starting point. Question 2 has students think of division using the missing-factor model.
Relating Multiplication and Division

How does an array show division?

In 1818, there were only 20 stars on the United States flag.

There were 4 equal rows of stars.

How many stars were in each row?

The array shows:

**How can a fact family help you divide?**

A fact family shows how multiplication and division are related.

Fact family for 4, 5, and 20:

\[
4 \times 5 = 20 \quad 20 \div 4 = 5 \\
5 \times 4 = 20 \quad 20 \div 5 = 4
\]

\[\text{factor} \times \text{factor} = \text{product} \quad \text{dividend} \div \text{divisor} = \text{quotient}\]

**Talk About It**

1. Skip count by 5s to find \(4 \times 5\). Then start at 20 and skip count by 5s backward to 0. The number of times you count back is the quotient for \(20 \div 5\).

2. How can you use the fact \(3 \times 6 = 18\) to find \(18 \div 3\)?

3. **Number Sense** Is \(3 \times 5 = 15\) part of the fact family for 3, 4, and 12? Explain.
Next consider how the four operations of addition, subtraction, multiplication, and division are related for the set of whole numbers. This is shown in Figure 3-32. Note that addition and subtraction are inverses of each other, as are multiplication and division with remainder 0. Also note that multiplication can be viewed as repeated addition, and division can be accomplished using repeated subtraction.

![Figure 3-32](image)

In Section 3-1, we have seen that the set of whole numbers is closed under addition and that addition is commutative and associative and has an identity. On the other hand, subtraction did not have these properties. In this section, we have seen that multiplication has some of the same properties that hold for addition. Does it follow that division behaves like subtraction? Investigate this in Now Try This 3-14.

**NOW TRY THIS 3-14**

a. Provide counterexamples to show that the set of whole numbers is not closed under division and that division is neither commutative nor associative.

b. Why is 1 not the identity for division?

---

**Division by 0 and 1**

Division by 0 and by 1 are frequently misunderstood by students. Before reading on, try to find the values of the following three expressions:

1. $3 \div 0$
2. $0 \div 3$
3. $0 \div 0$

Consider the following explanations:

1. By definition, $3 \div 0 = c$ if there is a unique whole number $c$ such that $0 \cdot c = 3$. Since the zero property of multiplication states that $0 \cdot c = 0$ for any whole number $c$, there is no whole number $c$ such that $0 \cdot c = 3$. Thus, $3 \div 0$ is undefined because there is no answer to the equivalent multiplication problem.

2. By definition, $0 \div 3 = c$ if there exists a unique whole number such that $3 \cdot c = 0$. Any number $c$ times 0 is 0, and in particular $3 \cdot 0 = 0$. Therefore, $c = 0$ and $0 \div 3 = 0$. Note that $c = 0$ is the only number that satisfies $3 \cdot c = 0$.

3. By definition, $0 \div 0 = c$ if there is a unique whole number $c$ such that $0 \cdot c = 0$. Notice that for any $c$, $0 \cdot c = 0$. According to the definition of division, $c$ must be unique. Since there is no unique number $c$ such that $0 \cdot c = 0$, it follows that $0 \div 0$ is undefined.
Dividing with 0 and 1

What are the division rules for 0 and 1?

**Example A**

Divide a number by 1.
\[ 4 \div 1 = ? \]
1 times what number = 4?
\[ 1 \times 4 = 4 \]
So, \[ 4 \div 1 = 4 \].

**Rule:** When any number is divided by 1, the quotient is that number.

**Example B**

Divide a number by itself.
\[ 7 \div 7 = ? \]
7 times what number = 7?
\[ 7 \times 1 = 7 \]
So, \[ 7 \div 7 = 1 \].

**Rule:** When any number (except 0) is divided by itself, the quotient is 1.

**Example C**

Divide zero by a number.
\[ 0 \div 2 = ? \]
2 times what number = 0?
\[ 2 \times 0 = 0 \]
So, \[ 0 \div 2 = 0 \].

**Rule:** When zero is divided by a number (except 0) the quotient is 0.

**Example D**

Divide a number by zero.
\[ 3 \div 0 = ? \]
0 times what number = 3?
There is no number that works, so, \[ 3 \div 0 \] cannot be done.

**Rule:** You cannot divide a number by 0.

**Talk About It**

1. How can you tell without dividing that \( 427 \div 1 = 427 \)?

Source: Scott Foresman-Addison Wesley Math, Grade 3, 2008 (p. 396).
Division involving 0 may be summarized as follows. Let \( n \) be any nonzero whole number. Then,

1. \( n \div 0 \) is undefined;  
2. \( 0 \div n = 0 \);  
3. \( 0 \div 0 \) is undefined.

Recall that \( n \cdot 1 = n \) for any whole number \( n \). Thus, by the definition of division, \( n \div 1 = n \). For example, \( 3 \div 1 = 3, 1 \div 1 = 1, \) and \( 0 \div 1 = 0 \). A grade 3 discussion of division by 0 and 1 can be found on the student page on page 155.

Order of Operations

Difficulties involving the order of arithmetic operations sometimes arise. For example, many students will treat \( 2 + 3 \cdot 6 \) as \( (2 + 3) \cdot 6 \), while others will treat it as \( 2 + (3 \cdot 6) \). In the first case, the value is 30; in the second case, the value is 20. To avoid confusion, mathematicians agree that when no parentheses are present, multiplications and divisions are performed before additions and subtractions. The multiplications and divisions are performed in the order they occur, and then the additions and subtractions are performed in the order they occur. Thus, \( 2 + 3 \cdot 6 = 2 + 18 = 20 \). This order of operations is not built into some calculators that display an incorrect answer of 30. The computation \( 8 - 9 \div 3 \cdot 2 + 3 \) is performed as

\[
8 - 9 \div 3 \cdot 2 + 3 = 8 - 3 \cdot 2 + 3 \\
= 8 - 6 + 3 \\
= 2 + 3 \\
= 5
\]

Assessment 3-3A

1. For each of the following, find, if possible, the whole numbers that make the equations true:
   a. \( 3 \cdot \Box = 15 \)
   b. \( 18 = 6 + 3 \cdot \Box \)
   c. \( \Box \cdot (5 + 6) = \Box \cdot 5 + \Box \cdot 6 \)

2. In terms of set theory, the product \( na \) could be thought of as the number of elements in the union of \( n \) sets with \( a \) elements in each. If this were the case, what must be true about the sets?

3. Determine if the following sets are closed under multiplication:
   a. \( \{0, 1\} \)
   b. \( \{2, 4, 6, 8, 10, \ldots\} \)
   c. \( \{1, 4, 7, 10, 13, \ldots\} \)

4. a. If 5 is removed from the set of whole numbers, is the set closed with respect to addition? Explain.
   b. If 5 is removed from the set of whole numbers, is the set closed with respect to multiplication? Explain.
   c. Answer the same questions as (a) and (b) if 6 is removed from the set of whole numbers.

5. Rename each of the following using the distributive property of multiplication over addition so that there are no parentheses in the final answer:
   a. \( (a + b)(c + d) \)
   b. \( \Box(\Delta + \Box) \)
   c. \( a(\Box + \epsilon) = ac \)

6. Place parentheses, if needed, to make each of the following equations true:
   a. \( 5 + 6 \cdot 3 = 33 \)
   b. \( 8 + 7 - 3 = 12 \)
   c. \( 6 + 8 - 2 \div 2 = 13 \)
   d. \( 9 + 6 \div 3 = 5 \)

7. Using the distributive property of multiplication over addition, we can factor as in \( x^2 + xy = x(x + y) \). Use the distributive property and other multiplication properties to factor each of the following:
   a. \( xy + y^2 \)
   b. \( xy + x \)
   c. \( a^2b + ab^2 \)

8. For each of the following, find whole numbers to make the statement true, if possible:
   a. \( 18 \div 3 = \Box \)
   b. \( \Box \div 76 = 0 \)
   c. \( 28 \div \Box = 7 \)

9. A sporting goods store has designs for six shirts, four pairs of pants, and three vests. How many different shirt-pants-vest outfits are possible?

10. Which property is illustrated in each of the following:
   a. \( 6(5 \cdot 4) = (6 \cdot 5)4 \)
11. Students are overheard making the following statements. What properties justify their statements?
   a. I know that 9·7 is either 63 or 69 and I know they can’t both be right.
   b. I know that 9·0 is 0 because I know that any number times 0 is 0.
   c. Any number times 1 is the same as the number we started with, so 9·1 is 9.

12. The product 6·14 can be found by thinking of the problem as $6(10 + 4) = (6·10) + (6·4) = 60 + 24 = 84$.
   a. What properties are being used?
   b. Use this technique to mentally compute 32·12.

13. The distributive property of multiplication over subtraction is $a(b - c) = ab - ac$
   Use this property to find each of the following:
   a. $9(10 - 2)$
   b. $20(8 - 3)$

14. Show that $(a + b)^2 = a^2 + 2ab + b^2$ using
   a. the distributive property of multiplication over addition and other properties.
   b. an area model.

15. If $a$ and $b$ are whole numbers with $a > b$, use the rectangles in the figure to explain why $(a + b)^2 - (a - b)^2 = 4ab$.

16. In each of the following, show that the left side of the equation is equal to the right side and give a reason for every step:
   a. $(ab)c = (ca)b$
   b. $(a + b)c = (c + a)b$

17. Factor each of the following:
   a. $xy - y^2$
   b. $47·101 - 47$
   c. $ab^2 - ba^2$

18. Rewrite each of the following division problems as a multiplication problem:
   a. $40 \div 8 = 5$
   b. $326 \div 2 = x$

19. Show that, in general, each of the following is false if $a$, $b$, and $c$ are whole numbers:
   a. $(a \div b) \div c = a \div (b \div c)$
   b. $a \div (b + c) = (a \div b) + (a \div c)$

20. Suppose $c$ is a divisor of $a$ and of $b$. Show that $(a + b) \div c = (a \div c) + (b \div c)$ using
   a. a model.
   b. the definition of division in terms of multiplication and the distributive property of multiplication over addition.

21. Find the solution for each of the following:
   a. $5x + 2 = 22$
   b. $3x + 7 = x + 13$
   c. $3(x + 4) = 18$

22. Millie and Samantha began saving money at the same time. Millie plans to save $\$3$ a month, and Samantha plans to save $\$5$ a month. After how many months will Samantha have saved exactly $\$10$ more than Millie?

23. There were 17 sandwiches for 7 people on a picnic. How many whole sandwiches were there for each person if they were divided equally? How many were left over?

24. a. Find all pairs of whole numbers whose product is 36.
   b. Plot the points found in (a) on a grid.
   c. Compare the pattern shape formed by the points to the graph of the pattern shape that could be found using all pairs of whole numbers whose sum is 36.

25. A new model of car is available in 4 exterior colors and 3 interior colors. Use a tree diagram and specific colors to show how many color schemes are possible for the car.

26. To find $7 \div 5$ on the calculator, press $7 \div 5 = \boxed{1.4}$, which yields 1.4. To find the whole-number remainder, ignore the decimal portion of 1.4, multiply 5·1, and subtract this product from 7. The result is the remainder. Use a calculator to find the whole-number remainder for each of the following divisions:
   a. $28 \div 5$
   b. $32 \div 10$
   c. $29 \div 3$
   d. $41 \div 7$
   e. $49,582 \div 14$

27. Is it possible to find a whole number less than 100 that when divided by 10 leaves remainder 4 and when divided by 47 leaves remainder 17?

28. In each of the following, tell what computation must be done last:
   a. $5(16 - 7) - 18$
   b. $54/(10 - 5 + 4)$
   c. $(14 - 3) + (24 + 2)$
   d. $21,045/345 + 8$

29. Write an algebraic expression for each of the following:
   a. Width of a rectangle whose area is $A$ and length is $l$
   b. $f$ feet in yards
   c. $b$ hours in minutes
   d. $d$ days in weeks
1. For each of the following, find, if possible, the whole numbers that make the equations true:
   a. \(8 \cdot □ = 24\)   b. \(28 = 4 + 6 \cdot □\)
   c. \(□ \cdot (8 + 6) = □ \cdot 8 + □ \cdot 6\)

2. Determine if the following sets are closed under multiplication:
   a. \(\{0\}\)   b. \(\{1, 3, 5, 7, 9, \ldots\}\)
   c. \(\{0, 1, 2\}\)

3. Rename each of the following using the distributive property of multiplication over addition so that there are no parentheses in the final answer. Simplify when possible.
   a. \(3(x + y + 5)\)
   b. \((x + y)(x + y + z)\)
   c. \(x(y + 1) - x\)

4. Place parentheses, if needed, to make each of the following equations true:
   a. \(4 + 3 \cdot 2 = 14\)
   b. \(9 + 3 + 1 = 4\)
   c. \(5 + 4 + 9 = 3 = 6\)
   d. \(3 + 6 - 2 + 1 = 7\)

5. The generalized distributive property for three terms states that for any whole numbers \(a\), \(b\), \(c\), and \(d\), \(a(b + c + d) = ab + ac + ad\). Justify this property using the distributive property for two terms.

6. Using the distributive property of multiplication over addition, we can factor as in \(x^2 + xy = x(x + y)\). Use the distributive property and other multiplication properties to factor each of the following:
   a. \(47 \cdot 99 + 47\)
   b. \((x + 1)y + (x + 1)\)
   c. \(x^2y + 5x^3\)

7. For each of the following, find whole numbers to make the statement true, if possible:
   a. \(27 \div 9 = □\)
   b. \(□ \div 52 = 1\)
   c. \(13 \div □ = 13\)

8. A new car comes in 5 exterior colors and 3 interior colors. How many different-looking cars are possible?

9. What multiplication is suggested by the following models?
   a. \[
   \begin{array}{cccccc}
   \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
   \end{array}
   \]
   b. \[
   \begin{array}{cccccc}
   \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
   \end{array}
   \]

10. Which property of whole numbers is illustrated in each of the following:
    a. \((5 \cdot 4)0 = 0\)
    b. \(7(3 \cdot 4) = 7(4 \cdot 3)\)
    c. \(7(3 \cdot 4) = (3 \cdot 4)7\)
    d. \((3 + 4)1 = 3 + 4\)
    e. \((3 + 4)5 = 3 \cdot 5 + 4 \cdot 5\)
    f. \((1 + 2)(3 + 4) = (1 + 2)3 + (1 + 2)4\)

11. Students are overheard making the following statements. What properties justify their statements?
   a. I know if I remember what 7 \(\cdot\) 9 is, then I also know what 9 \(\cdot\) 7 is.
   b. To find 9 \(\cdot\) 6, I just remember that 9 \(\cdot\) 5 is 45 and so 9 \(\cdot\) 6 is just 9 more than 45, or 54.

12. The product 5 \(\cdot\) 24 can be found by thinking of the problem as \(5(20 + 4) = 5 \cdot 20 + 5 \cdot 4 = 100 + 20 = 120\).
    a. What property is being used?
    b. Use this technique to mentally compute 8 \(\cdot\) 34.

13. The distributive property of multiplication over subtraction is
    \[a(b - c) = ab - ac\]
    Use this property to find each of the following:
    a. \((15(10 - 2)\)
    b. \(30(9 - 2)\)

14. Show that if \(b > c\), then \(a(b - c) = ab - ac\) using:
    a. an area model suggested by the given figure (express the shaded area in two different ways).

15. Show that the left-hand side of the equation is equal to the right-hand side and give a reason for every step.
    a. \((ab)c = b(ac)\)
    b. \(a(b + c) = ab + ac\)

16. Factor each of the following:
    a. \(xy - y\)
    b. \((x + 1)y - (x + 1)\)
    c. \(a^2b^3 - ab^2\)

17. Rewrite each of the following division problems as a multiplication problem:
    a. \(48 \div x = 16\)
    b. \(x \div 5 = 17\)

18. Think of a number. Multiply it by 2. Add 2. Divide by 2. Subtract 1. How does the result compare with your original number? Will this work all the time? Explain your answer.

19. Show that, in general, each of the following is false if \(a\), \(b\), and \(c\) are whole numbers:
    a. \(a \div b = b \div a\)
    b. \(a - b = b - a\)
20. Find the solution for each of the following:
   a. $5x + 8 = 28$  
   b. $5x + 6 = x + 14$  
   c. $5(x + 3) = 35$

21. String art is formed by connecting evenly spaced nails on the vertical and horizontal axes by segments of string. Connect the nail farthest from the origin on the vertical axis with the nail closest to the origin on the horizontal axis. Continue until all nails are connected, as shown in the figure that follows. How many intersection points are created with 10 nails on each axis?

22. Students were divided into eight teams with nine on each team. Later, the same students were divided into teams with six on each team. How many teams were there then?

23. Jonah has a large collection of marbles. He notices that if he borrows 5 marbles from a friend, he can arrange the marbles in rows of 13 each. What is the remainder when he divides his original number of marbles by 13?

24. In the following problems, use only the designated number keys on the calculator. You may use any function keys.

25. In each of the following, tell what computation must be done last:
   a. $5 \cdot 6 - 3 \cdot 4 + 2$
   b. $19 - 3 \cdot 4 + 9 \div 3$
   c. $15 - 6 + 2 \cdot 4$
   d. $5 + (8 - 2)3$

26. Find infinitely many whole numbers that leave remainder 3 upon division by 5.

27. The operation $\odot$ is defined on the set $S = \{a, b, c\}$, as shown in the following table. For example, $a \odot b = b$ and $b \odot a = b$.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$a$</td>
<td>$c$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
</tr>
</tbody>
</table>

   a. Is $S$ closed with respect to $\odot$?
   b. Is $\odot$ commutative on $S$?
   c. Is there an identity for $\odot$ on $S$? If yes, what is it?
   d. Try several examples to investigate the associative property for $\odot$ on $S$.

### Mathematical Connections 3-3

**Communication**

1. A number leaves remainder 6 when divided by 10. What is the remainder when the number is divided by 5? Justify your reasoning.
2. Can 0 be the identity for multiplication? Explain why or why not.
3. Suppose you forgot the product of 9 · 7. Give several ways that you could find the product using different multiplication facts and properties.
4. Is $x \div x$ always equal to 1? Explain your answer.
5. Is $x \cdot x$ ever equal to $x$? Explain your answer.
6. Describe all pairs of whole numbers whose sum and product are the same.

**Open-Ended**

7. Describe a real-life situation that could be represented by the expression $3 + 2 \cdot 6$.
8. How would you explain to a child that an even number has the form $2q$ and an odd number has the form $2q + 1$, where $q$ is a whole number?

### Cooperative Learning

9. Multiplication facts that most children have memorized can be stated in the table that is partially filled:

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\times & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & & & & & & & & & \\
2 & & & & & & & & & \\
3 & & & & & & & & & \\
4 & & & & & & & & & \\
5 & & & & & & & & & \\
6 & & & & & & & & & \\
7 & & & & & & & & & \\
8 & & & & & & & & & \\
9 & & & & & & & & & \\
\hline
\end{array}
$$
a. Fill out the table of multiplication facts. Find as many patterns as you can. List all the patterns that your group discovered and explain why some of those patterns occur in the table.
b. How can the multiplication table be used to solve division problems?
c. Consider the odd number 35 shown in the multiplication table. Consider all the numbers that surround it. Note that they are all even. Does this happen for all odd numbers in the table? Explain why or why not.

Questions from the Classroom
10. Suppose a student argued that $0 \div 0 = 0$ because “nothing divided by nothing” is “nothing.” How would you help that person?
11. Sue claims the following is true by the distributive law, where $a$ and $b$ are whole numbers:

$$3(ab) = (3a)(3b)$$

How might you help her?
12. a. A student claims that for all whole numbers $(ab) \div b = a$. How do you respond?
b. The student in part (a) claims that $0 \div 0 = 0$. The student's reasoning is, “If $a = 0$ and $b = 0$ are substituted in the equation in part (a), the result is $0 \cdot 0 \div 0 = 0$. But because $0 \cdot 0 = 0$, it follows that $0 \div 0 = 0$.” How do you respond?
13. A student asks if division on the set of whole numbers is distributive over subtraction. How do you respond?
14. A student says that 1 is the identity for division. How do you respond?

Review Problems
15. Give a set that is not closed under addition.
16. Is the operation of subtraction for whole numbers commutative? If not, give a counterexample.
17. What is wrong in each of the following?

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>137</td>
</tr>
<tr>
<td>$+56$</td>
<td>$+47$</td>
</tr>
<tr>
<td>183</td>
<td>712</td>
</tr>
</tbody>
</table>

Third International Mathematics and Science Study (TIMSS) Questions
In Toshi’s class there are twice as many girls as boys. There are 8 boys in the class. What is the total number of boys and girls in the class?

<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>b</td>
</tr>
<tr>
<td>c</td>
</tr>
<tr>
<td>d</td>
</tr>
</tbody>
</table>

TIMSS 2003, Grade 4
A piece of rope 204 cm long is cut into 4 equal pieces. Which of these gives the length of each piece in centimeters?

<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>b</td>
</tr>
<tr>
<td>c</td>
</tr>
<tr>
<td>d</td>
</tr>
</tbody>
</table>

TIMSS 2003, Grade 4
National Assessment of Educational Progress (NAEP) Question
The weights on the scale above are balanced. Each cube weighs 3 pounds. The cylinder weighs $N$ pounds. Which number sentence best describes this situation?

<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
</tr>
<tr>
<td>b</td>
</tr>
<tr>
<td>c</td>
</tr>
<tr>
<td>d</td>
</tr>
</tbody>
</table>

NAEP, 2007, Grade 4

LABORATORY ACTIVITY Enter a natural number less than 20 on the calculator. If the number is even, divide it by 2; if it is odd, multiply it by 3 and add 1. Next, use the number on the display. Follow the given directions. Repeat the process.

1. Will the display eventually reach 1?
2. Which number less than 20 takes the most steps before reaching 1?
3. Do even or odd numbers reach 1 more quickly?
4. Investigate what happens with numbers greater than 20.
In the grade 4 Focal Points, we find the following with respect to multiplication and division and the use of algorithms for doing computations:

Students use understandings of multiplication to develop quick recall of the basic multiplication facts and related division facts. They apply their understanding of models for multiplication (i.e., equal-sized groups, arrays, area models, equal intervals on the number line), place value, and properties of operations (in particular, the distributive property) as they develop, discuss, and use efficient, accurate, and generalizable methods to multiply multidigit whole numbers. They select appropriate methods and apply them accurately to estimate products or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. (p. 16)

In this section, multiplication and division algorithms will be developed using various models.

**Multiplication Algorithms**

To develop algorithms for multiplying multidigit whole numbers, we use the strategy of examining simpler computations first. Consider $4 \cdot 12$. This computation could be pictured as in Figure 3-33(a) with 4 rows of 12 blocks, or 48 blocks. These blocks in Figure 3-33(a) can also be partitioned to show that $4 \cdot 12 = 4(10 + 2) = 4 \cdot 10 + 4 \cdot 2$. The numbers $4 \cdot 10$ and $4 \cdot 2$ are partial products.

![Figure 3-33](image)

**Figure 3-33**

Figure 3-33(a) illustrates the distributive property of multiplication over addition on the set of whole numbers. The process leading to an algorithm for multiplying $4 \cdot 12$ is seen in Figure 3-33(b). Notice the similarity between the multiplication in Figure 3-33 and the following algebra multiplication:

$$4(x + 2) = 4x + 4 \cdot 2$$

$$= 4x + 8$$

Similarly, notice the analogy between the product

$$23 \cdot 14 = (2 \cdot 10 + 3)(1 \cdot 10 + 4)$$

and

$$(2x + 3)(1x + 4)$$
The analogy is continued as shown below:

\[
\begin{array}{c}
2 \cdot 10 + 3 \\
\times (1 \cdot 10 + 4)
\end{array} \quad \frac{2x + 3}{8x + 12} \\
\begin{array}{c}
8 \cdot 10 \\
3 \cdot 10
\end{array} \quad \frac{2x^2 + 3x}{2x^2 + 11x + 12}
\]

\[
2 \cdot 10^2 + 11 \cdot 10 + 12
\]

Multiplication of a three digit number by a one-digit factor will be explored after we discuss multiplication by a power of 10.

**Multiplication by 10\(^n\)**

Next we consider multiplication by powers of 10. First consider what happens when we multiply a given number by 10, such as 10 \(\times\) 23. If we start out with the base-ten block representation of 23, we have 2 longs and 3 units. To multiply by 10, we must replace each piece with a base-ten piece that represents the next higher power of 10. This is shown in Figure 3-34. Notice that the 3 units in 23 when multiplied by 10 become 3 longs or 3 tens. Therefore, after multiplication by 10 there are no units and hence we have 0 in the units place. In general, if we multiply any natural number by 10, we append a 0 at the end of the number.

![Figure 3-34](image-url)
To compute products such as $3 \cdot 200$, we proceed as follows:

\[
3 \cdot 200 = 3(2 \cdot 10^2) \\
= (3 \cdot 2)10^2 \\
= 6 \cdot 10^2 \\
= 6 \cdot 10^2 + 0 \cdot 10 + 0 \cdot 1 \\
= 600
\]

We see that multiplying $6$ by $10^2$ results in appending two zeros to the right of $6$. This idea can be generalized to the statement that multiplication of any natural number by $10^n$, where $n$ is a natural number, results in appending $n$ zeros to the right of the number.

**Remark** The appending of $n$ zeros to a natural number when multiplying by $10^n$ can also be explained as follows. We first multiply by 10, resulting in appending of one zero (as in $23 \cdot 10 = 230$). When we multiply by another 10, another zero is appended (as in $230 \cdot 10 = 2300$). Since we multiply $n$ times by 10, $n$ zeros are appended to the natural-number factor.

When multiplying powers of 10, the definition of exponents is used. For example, $10^2 \cdot 10^1 = (10 \cdot 10)10 = 10^3$, or $10^{2+1}$. In general, where $a$ is a natural number and $m$ and $n$ are whole numbers, $a^m \cdot a^n$ is given by the following:

\[
d^m \cdot a^n = \frac{(a \cdot a \cdot a \cdots \cdot a) \cdot (a \cdot a \cdot a \cdots \cdot a)}{m \text{ factors} \times n \text{ factors}} \\
= \frac{a \cdot a \cdot a \cdots \cdot a}{m + n \text{ factors}} = a^{m+n}
\]

Consequently, $a^m \cdot a^n = a^{m+n}$.

**Now Try This 3-15** Use the fact that $a^m \cdot a^n = a^{m+n}$ along other multiplication properties to explain why the computations in the cartoon are both true.

Multiplication by a power of 10 is helpful in calculating the product of a one-digit number and a three digit number. In the following example, we assume the previously developed
algorithm for multiplying a one-digit numeral times a two-digit numeral:

\[ 4 \cdot 367 = 4(3 \cdot 10^2 + 6 \cdot 10 + 7) \]
\[ = 4(3 \cdot 10^2) + 4(6 \cdot 10) + 4 \cdot 7 \]
\[ = (4 \cdot 3)10^2 + (4 \cdot 6)10 + 4 \cdot 7 \]
\[ = 1200 + 240 + 28 \]
\[ = 1468 \]

**NOW TRY THIS 3-16** Use expanded notation and an approach similar to the preceding to calculate 7 \cdot 4589.

**Multiplication with Two-Digit Factors**

Consider 14 \cdot 23. Model this computation by first using base-ten blocks, as shown in Figure 3-35(a), and then showing all the partial products and adding, as shown in Figure 3-35(b).

![Figure 3-35](image_url)

This last approach leads to an algorithm for multiplication:

\[
\begin{array}{c}
23 \\
\times 14 \\
\hline
92 \\
(4 \cdot 23) \\
230 \\
\hline
322 \\
\end{array} \quad \begin{array}{c}
23 \\
\times 14 \\
\hline
92 \\
(10 \cdot 4) \\
230 \\
\hline
322 \\
\end{array}
\]

We are accustomed to seeing the partial product 230 written without the zero, as 23. The placement of 23 with 3 in the tens column obviates having to write the 0 in the units column. But when children are first learning multiplication algorithms, we should encourage them to include the zero. This promotes better understanding and helps to avoid errors.

The distributive property of multiplication over addition can be used to explain why the algorithm for multiplication works. Again, consider 14 \cdot 23.

\[
14 \cdot 23 = (10 + 4)23 \\
= 10 \cdot 23 + 4 \cdot 23 \\
= 230 + 92 \\
= 322
\]
Because algorithms are powerful, there is sometimes a tendency to overapply them or to use paper and pencil for a task that should be done mentally. For example, consider

\[
\begin{array}{c}
\phantom{0}213 \\
\times\phantom{0}1000 \\
\hline
\phantom{0}000 \\
\phantom{0}000 \\
\phantom{0}213 \\
\hline
\phantom{0}213000
\end{array}
\]

This application is not wrong but is inefficient. Mental math and estimation are important skills in learning mathematics and should be practiced in addition to paper-and-pencil computations. Children should be encouraged to estimate whether their answers are reasonable.

In the computation \(14 \cdot 23\), we know that the answer must be between \(10 \cdot 20 = 200\) and \(20 \cdot 30 = 600\) because \(10 < 14 < 20\) and \(20 < 23 < 30\).

**Lattice Multiplication**

Lattice multiplication has the advantage of delaying all additions until the single-digit multiplications are complete. Because of this, it is sometimes referred to as a “low-stress algorithm.” Low-achieving students especially seem to like this algorithm, perhaps because of the structure provided by the lattice. The lattice multiplication algorithm for multiplying 14 and 23 is shown in Figure 3-36. (Determining the reasons why lattice multiplication works is left as an exercise.)

![Figure 3-36](image)

Lattice multiplication dates back to tenth-century India. This algorithm was imported to Europe and was popular in the fourteenth and fifteenth centuries. Napier’s rods (or bones), developed by John Napier in the early 1600s, were modeled on lattice multiplication. Napier’s rods can be used in a multiplication procedure.
Division Algorithms

Using Repeated Subtraction to Develop the Standard Division Algorithm

One algorithm for division of whole numbers was developed in an earlier section using repeated subtraction. However, it could have been done more efficiently. Consider the following:

A shopkeeper is packaging juice in cartons that hold 6 bottles each. She has 726 bottles. How many cartons does she need?

We might reason that if 1 carton holds 6 bottles, then 10 cartons hold 60 bottles and 100 cartons hold 600 bottles. If 100 cartons are filled, there are $726 - 100 \cdot 6$, or 126, bottles remaining. If 10 more cartons are filled, then $126 - 10 \cdot 6$, or 66, bottles remain. Similarly, if 10 more cartons are filled, $66 - 10 \cdot 6$, or 6, bottles remain. Finally, 1 carton will hold the remaining 6 bottles. The total number of cartons necessary is $100 + 10 + 10 + 1$, or 121. This procedure is summarized in Figure 3-37(a). A more efficient method is shown in Figure 3-37(b).

We might reason that if 1 carton holds 6 bottles, then 10 cartons hold 60 bottles and 100 cartons hold 600 bottles. If 100 cartons are filled, there are $726 - 100 \cdot 6$, or 126, bottles remaining. If 10 more cartons are filled, then $126 - 10 \cdot 6$, or 66, bottles remain. Similarly, if 10 more cartons are filled, $66 - 10 \cdot 6$, or 6, bottles remain. Finally, 1 carton will hold the remaining 6 bottles. The total number of cartons necessary is $100 + 10 + 10 + 1$, or 121. This procedure is summarized in Figure 3-37(a). A more efficient method is shown in Figure 3-37(b).

Divisions such as the one in Figure 3-38 are usually shown in elementary school texts in the most efficient form, as in Figure 3-38(b), in which the numbers in color in Figure 3-38(a) are omitted. The technique used in Figure 3-38(a) is often called
“scaffolding” and may be used as a preliminary step to achieving the standard algorithm, as in Figure 3-38(b). Notice that scaffolding takes the numbers on the right in Figure 3-37(b), and places them on the top as in Figure 3-38(a). Note that the scaffolding shows place value and, as indicated in the Research Note, place value is important to understanding the standard algorithms.

**Using Base-Ten Blocks to Develop the Standard Division Algorithm**

As pointed out in the Research Note, students need to see why each move in an algorithm is appropriate rather than just what sequence of moves to make. Next we use base-ten blocks to justify why each move in the standard algorithm is appropriate. In Table 3-4, the base-ten model is on the left with the corresponding steps in the standard algorithm on the right.

Students constructing meanings underlying an operation such as long division need to focus on understanding why each move in an algorithm is appropriate rather than on which moves to make and in which sequence. Also, teachers should encourage students to invent their own personal procedures for the operations but expect them to explain why their inventions are legitimate (Lampert 1992).

**Research Note**

**Table 3-4**

<table>
<thead>
<tr>
<th>Base-Ten Blocks</th>
<th>Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. First we represent 726 with base-ten blocks.</td>
<td>$6 \cdot 726$</td>
</tr>
<tr>
<td>2. We next determine how many sets of 6 flats (hundreds) there are in the representation. There is 1 set of 6 flats with 1 flat, 2 longs (tens), and 6 units (ones) left over.</td>
<td>1 set of 6 flats</td>
</tr>
<tr>
<td></td>
<td>$6 \cdot 726$</td>
</tr>
<tr>
<td></td>
<td>$- 6$</td>
</tr>
<tr>
<td></td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>1 flat</td>
</tr>
<tr>
<td></td>
<td>2 longs</td>
</tr>
<tr>
<td></td>
<td>6 units left over</td>
</tr>
<tr>
<td>3. Next, we convert the one leftover flat to 10 longs (tens). Now we have 12 longs (tens) and 6 units (ones).</td>
<td>1 set of 6 flats</td>
</tr>
<tr>
<td></td>
<td>$6 \cdot 726$</td>
</tr>
<tr>
<td></td>
<td>$- 6$</td>
</tr>
<tr>
<td></td>
<td>$12$</td>
</tr>
<tr>
<td></td>
<td>12 longs</td>
</tr>
<tr>
<td></td>
<td>6 units left over</td>
</tr>
</tbody>
</table>

*(continued)*
Whole Numbers and Their Operations

4. Next we determine how many sets of 6 longs (tens) there are in 12 longs and 6 units. We have 2 sets of 6 longs and 6 units left over.

5. Next we determine how many sets of 6 units (ones) there are in the 6 remaining units. There is 1 set of 6 units with no units left over (the remainder is 0).

Therefore, we see that in the base-ten block representation of 726, there is 1 group of 6 flats (hundreds), 2 groups of 6 longs (tens), and 1 group of 6 units (ones) with none left over. Hence, the quotient is 121 with a remainder of 0. The steps in the algorithm are shown alongside the work with the base-ten blocks.

Short Division

The process used in Figure 3-38(b) is usually referred to as “long” division. Another technique, called “short” division, can be used when the divisor is a one-digit number and most of the work is done mentally. An example of the short division algorithm is given in Figure 3-39.

Figure 3-39

Division in many elementary texts is taught using a four-step algorithm: estimate, multiply, subtract, and compare. This is demonstrated in the student page on page 170. Notice that students check the division by using the inverse operation of multiplication. Study the student page and answer the question at the bottom of the page.

**Division by a Two-Digit Divisor**

An example of division by a divisor of more than one digit is given next. Consider 32\( \div \)2618.

1. Estimate the quotient in 32\( \div \)2618. Because 1 \( \cdot \) 32 = 32, 10 \( \cdot \) 32 = 320, 100 \( \cdot \) 32 = 3200, we see that the quotient is between 10 and 100.

2. Find the number of tens in the quotient. Because 26 \( \div \) 3 is approximately 8, 26 hundreds divided by 3 tens is approximately 8 tens. We then write the 8 in the tens place, as shown:

\[
\begin{array}{cccc}

& & 8 & 0 \\
32 & \cancel{2} & 6 & 18 \\
\hline
\end{array}
\]

\[
\begin{array}{cccc}
80 & & & \\
2560 & & & \\
\hline
58 & & & \\
80 & \cancel{2} & 6 & 18 \\
\hline
\end{array}
\]

3. Find the number of units in the quotient. Because 5 \( \div \) 3 is approximately 1, 5 tens divided by 3 tens is approximately 1. This is shown on the left, with the standard algorithm shown on the right.

\[
\begin{array}{cccc}

& & 8 & 1 \\
32 & \cancel{2} & 6 & 18 \\
\hline
80 & & & \\
256 & & & \\
\hline
58 & & & \\
\end{array}
\]

\[
\begin{array}{cccc}

& & 8 & 1 R 26 \\
32 & \cancel{2} & 6 & 18 \\
\hline
26 & & & \\
\end{array}
\]

4. Check: 32 \( \cdot \) 81 + 26 = 2618.

Normally in grade-school books, we see the format shown on the right, which places the remainder beside the quotient.

**Multiplication and Division in Different Bases**

In multiplication, as with addition and subtraction, we need to identify the basic facts of single-digit multiplication before we can develop any algorithms. The multiplication facts for base five are given in Table 3-5. These facts can be derived by using repeated addition.

**Table 3-5 Base-Five Multiplication Table**

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>11</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>11</td>
<td>14</td>
<td>22</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>4</td>
<td>13</td>
<td>22</td>
<td>31</td>
</tr>
</tbody>
</table>
Dividing Three-Digit Numbers

How do you divide larger numbers?

A school bus company has 273 buses and five parking lots. If the company wants to park the same number of buses in each lot, how many buses should be parked in each lot?

Since the company wants to park the same number of buses in each lot, you can divide.

Example A

Find 273 ÷ 5.

Estimate: 273 is close to 250 and 250 ÷ 5 = 50, so the quotient is a little more than 50.

\[
\begin{array}{c|c|c|c}
\text{STEP 1} & \text{STEP 2} & \text{CHECK} \\
\hline
\text{Divide the tens.} & \text{Bring down the ones and divide.} & \text{Multiply the quotient by the divisor and add the remainder.} \\
5 & 54 & 2 \\
\frac{273}{5} & \frac{273}{5} & 273 \\
\frac{25}{2} & \frac{25}{2} & 3 \\
\frac{2}{2} & \frac{2}{2} & \frac{3}{2} \\
\end{array}
\]

So, \(273 ÷ 5 = 54 \text{ R}3\).

The company can park 54 buses in each lot. They will have 3 buses left over.

Talk About It

1. In Example A, why do you start dividing with the tens?
There are various ways to do the multiplication $21_{\text{five}} \cdot 3_{\text{five}}$:

\[
\begin{array}{c|c}
\text{Fives} & \text{Ones} \\
\hline
2 & 1 \\
\times & 3 \\
\hline
110 & (10 + 3)_{\text{five}} \\
113 & (110 + 3)_{\text{five}} \rightarrow 110 \\
\end{array}
\]

The multiplication of a two-digit number by a two-digit number is developed next:

\[
\begin{array}{c}
23_{\text{five}} \\
\times 14_{\text{five}} \\
\hline
22 & (4 \cdot 3)_{\text{five}} \\
130 & (4 \cdot 20)_{\text{five}} \\
30 & (10 \cdot 3)_{\text{five}} \\
200 & (10 \cdot 20)_{\text{five}} \\
\hline
230 & 432_{\text{five}} \\
\end{array}
\]

Lattice multiplication can also be used to multiply numbers in various number bases. This is explored in Assessment 3-4.

Division in different bases can be performed using the multiplication facts and the definition of division. For example, $22_{\text{five}} \div 3_{\text{five}} = c$ if, and only if, $c \cdot 3_{\text{five}} = 22_{\text{five}}$. From Table 3-5, we see that $c = 4_{\text{five}}$. As in base ten, computing multidigit divisions efficiently in different bases requires practice. The ideas behind the algorithms for division can be developed by using repeated subtraction. For example, $3241_{\text{five}} \div 43_{\text{five}}$ is computed by means of the repeated-subtraction technique in Figure 3-40(a) and by means of the conventional algorithm in Figure 3-40(b). Thus, $3241_{\text{five}} \div 43_{\text{five}} = 34_{\text{five}}$ with remainder $14_{\text{five}}$.

\[
\begin{array}{c|c}
\text{(a)} & \text{(b)} \\
\hline
43_{\text{five}} & 43_{\text{five}} \\
-430 & 34_{\text{five}} \overline{R14_{\text{five}}} \\
2311 & -234 \\
-430 & -401 \\
1331 & -332 \\
-430 & 14_{\text{five}} \\
401 & \text{\ldots} \\
-141 & \text{\ldots} \\
-210 & \text{\ldots} \\
-141 & \text{\ldots} \\
14 & \text{\ldots} \\
\end{array}
\]

**Figure 3-40**

Computations involving base two are demonstrated in Example 3-4.

**Example 3-4**

a. Multiply:

\[
101_{\text{two}} \times 11_{\text{two}}
\]

b. Divide:

\[
101_{\text{two}} \overline{\div 110110_{\text{two}}}
\]
Assessment 3-4A

1. Fill in the missing numbers in each of the following:
   a. \( \frac{4.6}{783} \times 1.78 \)  
   b. \( \frac{327}{9.1} \times 3.08 \)

2. Perform the following multiplications using the lattice multiplication algorithm:
   a. \( \frac{728}{94} \times 306 \)
   b. \( \frac{306}{24} \times 306 \)

3. Explain why the lattice multiplication algorithm works.

4. Simplify each of the following using properties of exponents. Leave answers as powers.
   a. \( 5^7 \cdot 5^{12} \)
   b. \( 6^{10} \cdot 6^2 \cdot 6^3 \)
   c. \( 10^{296} \cdot 10^{17} \)
   d. \( 2^7 \cdot 10^5 \cdot 5^7 \)

5. a. Which is greater, \( 2^{80} \) or \( 2^{100} \)? Why?
   b. Which is greatest, \( 2^{101} \), \( 3 \cdot 2^{100} \), or \( 2^{102} \)? Why?

6. The following model illustrates \( 22 \cdot 13 \):

   ![Model Illustrating 22 x 13]

   a. Explain how the partial products are shown in the figure.
   b. Draw a similar model for \( 15 \cdot 21 \).
   c. Draw a similar base-five model for the product \( 43_{\text{five}} \cdot 23_{\text{five}} \). Explain how the model can be used to find the answer in base five.

7. Consider the following:
   \( \frac{476}{952} \times 293 \)  
   \( \frac{4284}{1428} = (2 \cdot 476) \)
   \( \frac{139468}{139468} = (3 \cdot 476) \)

   a. Use the conventional algorithm to show that the answer is correct.
   b. Explain why the algorithm works.
   c. Try the method to multiply \( 84 \times 363 \).

8. The Russian peasant algorithm for multiplying \( 27 \times 68 \) follows. (Disregard remainders when halving.)

   \[
   \begin{array}{c|c|c}
   \text{Halves} & \text{Doubles} \\
   \hline
   27 & 68 \\
   \hline
   13 & 136 \\
   6 & 272 \\
   3 & 544 \\
   \end{array}
   \]

   In the “Halves” column, choose the odd numbers. In the “Doubles” column, circle the numbers paired with the odds from the “Halves” column. Add the circled numbers.

   \[
   \begin{array}{c|c|c|c}
   68 & 136 & 272 & 544 \\
   \hline
   1088 \hspace{1cm} & 1836 \hspace{1cm} & \text{This is the product of 27 \times 68.} \\
   \end{array}
   \]

   Try this algorithm for \( 17 \cdot 63 \) and other numbers.

9. Answer the following questions based on the activity chart given next:
a. How many calories are burned during 3 hr of cross-country skiing?
b. Jane played tennis for 2 hr while Carolyn played volleyball for 3 hr. Who burned more calories, and how many more?
c. Lyle went snowshoeing for 3 hr and Maurice went cross-country skiing for 5 hr. Who burned more calories, and how many more?

10. On a 14-day vacation, Glenn increased his caloric intake by 1500 calories per day. He also worked out more than usual by swimming 2 hr a day. Swimming burns 666 calories per hour, and a net gain of 3500 calories adds 1 lb of weight. Did Glenn gain at least 1 lb during his vacation?

11. Perform each of the following divisions using both the repeated-subtraction and standard algorithms:
   a. 35 × 26
   b. 5 3
      90
      15
      15
      0

12. Using a calculator, Ralph multiplied by 10 when he should have divided by 10. The display read 300. What should the correct answer be?

13. The following figure shows four operation machines. The output from one machine becomes the input for the one below it. Complete the accompanying chart.

14. Consider the following multiplications. Notice that when the digits in the factors are reversed, the products are the same.

   36 × 42 = 63 × 24 = 1512

   a. Find other multiplications where this procedure works.
   b. Find a pattern for the numbers that work in this way.

15. Molly read 160 pages in her book in 4 hr. Her sister Karly took 4 hr to read 100 pages in the same book. If the book is 200 pages long and if the two girls continued to read at these rates, how much longer would it take Karly to read the book than Molly?

16. Dan has 4520 pennies in three boxes. He says that there are 3 times as many pennies in the first box as in the third and twice as many in the second box as in the first. How much does he have in each box?

17. Gina buys apples from an orchard and then sells them at a country fair in bags of 3 for $1 a bag. She bought 50 boxes of apples, 36 apples in a box, and paid $452. If she sold all but 18 apples, what was her total profit?

18. Discuss possible error patterns in each of the following:
   a. 36 × 7
   b. 13 ÷ 4

19. a. Give reasons for each of the following steps:
   
   \[ 56 \cdot 10 = (5 \cdot 10 + 6) \cdot 10 = 5 \cdot 100 + 6 \cdot 10 = 5 \cdot 10^2 + 6 \cdot 10 = 5 \cdot 10^3 + 6 \cdot 10 + 0 \cdot 1 = 560 \]

   b. Give reasons for each step in computing $34 \cdot 10^2$.

20. To transport the complete student body of 1672 students to a talk given by the governor, the school plans to rent buses that can hold 29 students each. How many buses are needed? Will all the buses be full?

21. Place the digits 7, 6, 8, and 3 in the boxes to obtain

\[ \begin{array}{c}
\text{Input} & \text{Output} \\
2 & 11 \\
4 & 0 \\
0 & 19 \\
3 & 31 \\
\end{array} \]

   a. the greatest product.
   b. the least product.

22. For what possible bases are each of the following computations correct?
   a. 213 + 308 = 522
   b. 213 × 32 = 430
   c. 1043 \div 11300

23. a. Use lattice multiplication to compute $32_5 \cdot 4_5$.
   b. Find the smallest values of $a$ and $b$ such that $32_a = 23_b$.

24. Perform each of these operations using the bases shown:
   a. $32_5 \cdot 4_5$
   b. $32_5 \div 4_5$
   c. $43_{10} \cdot 23_{10}$
   d. $143_5 \div 3_5$
   e. $10010_{two} \div 11_{two}$
   f. $10110_{two} \cdot 101_{two}$
1. Fill in the missing numbers in the following:

\[
\begin{array}{c}
4.4 \\
\times 327 \\
\hline
3.88 \\
9.68 \\
\hline
452 \\
1582.8
\end{array}
\]

2. Perform the following multiplications using the lattice multiplication algorithm:
   a. \(327 \times 43\)
   b. \(2618 \times 137\)

3. The following chart gives average water usage for one person for one day:

<table>
<thead>
<tr>
<th>Use</th>
<th>Average Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taking bath</td>
<td>110 L (liters)</td>
</tr>
<tr>
<td>Taking shower</td>
<td>75 L</td>
</tr>
<tr>
<td>Flushing toilet</td>
<td>22 L</td>
</tr>
<tr>
<td>Washing hands, face</td>
<td>7 L</td>
</tr>
<tr>
<td>Getting a drink</td>
<td>1 L</td>
</tr>
<tr>
<td>Brushing teeth</td>
<td>1 L</td>
</tr>
<tr>
<td>Doing dishes (one meal)</td>
<td>30 L</td>
</tr>
<tr>
<td>Cooking (one meal)</td>
<td>18 L</td>
</tr>
</tbody>
</table>

   a. Use the chart to calculate how much water you use each day.

   b. The average American uses approximately 200 L of water per day. Are you average?

   c. If there are 310,000,000 people in the United States, on average approximately how much water is used in the United States per day?

4. Simplify each of the following using properties of exponents. Leave answers as powers.
   a. \(3^3 \cdot 3^4\)
   b. \(5^2 \cdot 5^4 \cdot 5^2\)
   c. \(6^2 \cdot 2^2 \cdot 3^2\)

5. a. Which is greater, \(2^{20} + 2^{20}\) or \(2^{21}\)? Why?
    b. Which is greatest, \(3^{31}, 9 \cdot 3^{30}\), or \(3^{33}\)? Why?

6. The following model illustrates \(13 \cdot 12\):

   a. Explain how the partial products are shown in the figure.
   b. Draw a similar model for \(12 \cdot 22\).

7. a. Use base-five blocks to compute \(14_5 \cdot 23_5\).
    b. Use the distributive property of multiplication over addition to explain why multiplication of a natural number in base five by \(10_5\) results in annexation of 0 to the number.
    c. Explain why multiplication of a natural number in base five by \(100_5\) results in annexation of two 0s to the number.
    d. Use the distributive property of multiplication over addition and part (b) to compute \(14_5 \cdot 23_5\).

8. Complete the following table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(a \cdot b)</th>
<th>(a + b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>56</td>
<td>3752</td>
<td>110</td>
</tr>
<tr>
<td></td>
<td></td>
<td>270</td>
<td>33</td>
</tr>
</tbody>
</table>

9. Sue purchased a $30,000 life-insurance policy at the price of $24 for each $1000 of coverage. If she pays the premium in 12 monthly installments, how much is each installment?

10. Perform each of the following divisions using both the repeated-subtraction and the standard algorithms:
    a. \(7 \overline{)392}\)
    b. \(37 \overline{)925}\)
    c. \(423 \overline{)5002}\)

11. Place the digits 7, 6, 8, and 3 in the boxes \(\square \square \square \square\) to obtain
    a. the greatest quotient.
    b. the least quotient.

12. Twenty members of the band plan to attend a festival. The band members washed 245 cars at $2 per car to help cover expenses. The school will match every dollar the band raises with a dollar from the school budget. The cost of renting the bus to take the band is $72 per mile and the round-trip is 350 mi. The band members can stay in the dorm for two nights at $5 per person per night. Meals for the trip will cost $28 per person. Has the band raised enough money yet? If not, how many more cars do they have to wash?

13. The following figure shows three operation machines. The output from one machine becomes the input for the one below it. Complete the accompanying chart.
14. Choose three different digits.
   a. Create all possible two-digit numbers from the numbers you chose. Each number can be used only once.
   b. Add the six numbers.
   c. Add the three digits you chose.
   d. Divide the answer in (b) by the answer in (c).
   e. Repeat (a) through (d) with three different digits.
   f. Is the final result always the same? Why?

15. Xuan saved $5340 in 3 years. If he saved $95 per month in the first year and a fixed amount per month for the next 2 years, how much did he save per month during the last 2 years?

16. A group of fourth-grade children had to cut four pieces of ribbon each 4 ft long from a ribbon of 44 yd. What is the length of the remaining ribbon?

17. Discuss possible error patterns in each of the following:
   a. $34 \times 8$
   b. $34 \times 6$
   c. $2432 \div 11$

18. Give reasons for each of the following steps:
   \[
   35 \times 100 = (3 \times 10 + 5)100 = (3 \times 10 + 5)10^2 = (3 \times 10 + 5)10^2 + 5 \times 10^2 = 3 \times 10^3 + 5 \times 10^2 = 3 \times 10^3 + 5 \times 10^2 + 0 \times 10 + 0 \times 1 = 3500
   \]

19. a. Find all the whole numbers that leave remainder 1 upon division by 4. Write your answer using set builder notation.
   b. Write the numbers from part (a) in a sequence starting from the smallest.
   c. What kind of sequence is the one in part (b)?

20. Perform each of these operations using the bases shown:
   a. $42_{\text{five}} \cdot 3_{\text{five}}$
   b. $22_{\text{five}} \div 4_{\text{five}}$
   c. $32_{\text{five}} \cdot 42_{\text{five}}$
   d. $1313_{\text{five}} \div 23_{\text{five}}$
   e. $101_{\text{two}} \cdot 101_{\text{two}}$
   f. $1001_{\text{two}} \div 11_{\text{two}}$

21. For what possible bases are each of the following computations correct?
   a. $322 \div 23 = 11_{11}$
   b. $101 \div 11 = 11_{11}$
   c. $101 \div 11 = 0_{11}$

22. a. Use lattice multiplication to compute $423_{\text{five}} \cdot 23_{\text{five}}$
   b. Find the smallest values of $a$ and $b$ such that $41^a = 14^b$.

23. Place the digits 7, 6, 8, 3, and 2 in the boxes to obtain
   \[
   \begin{array}{cc}
   \text{7} & \text{6} \\
   \text{3} & \text{2}
   \end{array}
   \times
   \begin{array}{cc}
   \text{8} & \text{5} \\
   \text{2} & \text{1}
   \end{array}
   \]
   a. the greatest product.
   b. the least product.

24. Find the products of the following and describe the pattern that emerges:
   a. $1 \times 1$
   b. $99 \times 99$
   c. $11 \times 11$
   d. $111 \times 111$
   e. $1111 \times 1111$
   f. $1001_{\text{two}} \cdot 101_{\text{two}}$
   g. $101_{\text{two}} \div 11_{\text{two}}$
   h. $101_{\text{two}} \div 11_{\text{two}}$
   i. $1001_{\text{two}} \div 11_{\text{two}}$
   j. $101_{\text{two}} \div 11_{\text{two}}$
   k. $101_{\text{two}} \div 11_{\text{two}}$
   l. $1001_{\text{two}} \div 11_{\text{two}}$
   m. $101_{\text{two}} \div 11_{\text{two}}$
   n. $101_{\text{two}} \div 11_{\text{two}}$
   o. $1001_{\text{two}} \div 11_{\text{two}}$
   p. $101_{\text{two}} \div 11_{\text{two}}$
   q. $101_{\text{two}} \div 11_{\text{two}}$
   r. $1001_{\text{two}} \div 11_{\text{two}}$
   s. $101_{\text{two}} \div 11_{\text{two}}$
   t. $101_{\text{two}} \div 11_{\text{two}}$
   u. $1001_{\text{two}} \div 11_{\text{two}}$
   v. $101_{\text{two}} \div 11_{\text{two}}$
   w. $101_{\text{two}} \div 11_{\text{two}}$
   x. $1001_{\text{two}} \div 11_{\text{two}}$
   y. $101_{\text{two}} \div 11_{\text{two}}$
   z. $101_{\text{two}} \div 11_{\text{two}}$

25. Test the patterns discovered. If the patterns do not continue as expected, determine when the patterns stop.

Mathematical Connections 3-4

Communication

1. How would you explain to children how to multiply 345·678, assuming that they know and understand multiplication by a single digit and multiplication by a power of 10?

2. What happens when you multiply any two-digit number by 101? Explain why this happens.


4. Do you think it is valuable for students to see more than one method of doing computation problems? Why or why not?

5. Choose what you consider the “best” algorithm studied in this section. Explain the reasoning behind your choice.

6. Tom claims that long division should receive reduced attention in elementary classrooms. Do you agree or disagree? Defend your answer.

7. Prove that all numbers of the form \(abba\) (where \(a\) and \(b\) are digits in base ten) leave remainder 0 upon division by 11.
Is the same true for all the numbers of the form $abccba$? Why or why not?

Open-Ended
8. If a student presented a new “algorithm” for computing with whole numbers, describe the process you would recommend to the student to determine if the algorithm would always work.

Cooperative Learning
9. The traditional sequence for teaching operations in the elementary school is first addition, then subtraction, followed by multiplication, and finally division. Some educators advocate teaching addition followed by multiplication, then subtraction followed by division. Within your group prepare arguments for teaching the operations in either order listed.

Questions from the Classroom
10. A student divides as follows. How would you help?

```
  45
3)1215
-12
  15
- 15
  0
```

11. A student divides as follows. How do you help?

```
  15
6)36
- 6
 30
- 30
```

12. A student asks how you can find the quotient and the remainder in a division problem like $592 \div 36$ using a calculator without an integer division button.

13. A student claims that to divide a number with the units digit 0 by 10, she just crosses out the 0 to get the answer. She wants to know if this is always true and why and if the 0 has to be the units digit. How do you respond?

14. A student claims that if the remainder when $m$ is divided by $n$ is 0, then the dividend ($m$) and the divisor ($n$) can each be multiplied by the same nonzero whole number $c$ and the answer to the division stays the same. That is, $m \div n = (mc) \div (nc)$. She wants to know why. How would you respond, assuming the student does not know anything about fractions?

15. a. A student asks if $39 + 41 = 40 + 40$, is it true that $39 \cdot 41 = 40 \cdot 40$. How do you reply?

b. Another student says that he knows that $39 \cdot 41 \neq 40 \cdot 40$ but he found that $39 \cdot 41 = 40 \cdot 40 - 1$. He also found that $49 \cdot 51 = 50 \cdot 50 - 1$. He wants to know if this pattern continues. How would you respond?

Review Problems
16. Illustrate the identity property of addition for whole numbers.

17. Rename each of the following using the distributive property of multiplication over addition:
   a. $ax + bx + 2x$
   b. $3(a + b) + x(a + b)$

18. At the beginning of a trip, the odometer registered 52,281. At the end of the trip, the odometer registered 59,260. How many miles were traveled on this trip?

19. Write each of the following division problems as a multiplication problem:
   a. $36 \div 4 = 9$
   b. $112 \div 2 = x$
   c. $48 \div x = 6$
   d. $x \div 7 = 17$

Third International Mathematics and Science Study (TIMSS) Question
Each student needs 8 notebooks for school. How many notebooks are needed for 115 students?

Use the tiles $\blacksquare$, $\blacksquare$, and $\blacksquare$. Write the numbers on the tiles in the boxes below to make the largest answer when you multiply.

```
\blacksquare \times \blacksquare
```

$37 \times \blacksquare = 703$.

What is the value of $37 \times \blacksquare + 6$?

TIMSS, 2003, Grade 4

National Assessment of Educational Progress (NAEP) Question
There will be 58 people at a breakfast and each person will eat 2 eggs. There are 12 eggs in each carton. How many cartons of eggs will be needed for the breakfast?

a. 9
b. 10
c. 72
d. 116

NAEP 2007, Grade 4
**BRAIN TEASER** For each of the following, replace the letters with digits in such a way that the computation is correct. Each letter may represent only one digit.

a. LYNDON

\[ \times B \quad MA \]

\[ \quad JOHNSON \quad + \quad MA \]

\[ = \quad EEL \]

d. LABORATORY ACTIVITY

1. Messages can be coded on paper tape in base two. A hole in the tape represents 1, whereas the absence of a space represents 0. The value of each hole depends on its position; from left to right, 16, 8, 4, 2, 1 (all powers of 2). Letters of the alphabet may be coded in base two according to their position in the alphabet. For example, G is the seventh letter. Since \( 7 = 1 \cdot 4 + 1 \cdot 2 + 1 \), the holes appear as they do in Figure 3-41:

\[ \begin{array}{cccc}
16 & 8 & 4 & 2 \\
1 & & & 1
\end{array} \]

![Figure 3-41](image)

a. Decode the message in Figure 3-42.

b. Write your name on a tape using base two.

2. Consider the cards in Figure 3-43 that are modeled on base-two arithmetic.

<table>
<thead>
<tr>
<th>Card E</th>
<th>Card D</th>
<th>Card C</th>
<th>Card B</th>
<th>Card A</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>24</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>25</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>26</td>
<td>6</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>19</td>
<td>27</td>
<td>7</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>20</td>
<td>28</td>
<td>8</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>21</td>
<td>29</td>
<td>9</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>22</td>
<td>30</td>
<td>10</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>23</td>
<td>31</td>
<td>11</td>
<td>9</td>
<td>15</td>
</tr>
</tbody>
</table>

![Figure 3-43](image)

a. Suppose a person’s age appears on cards E, C, and B, and the person is 22. Can you discover how this works and why?

b. Design card F and redesign cards A to E so that the numbers 1 through 63 can be used.
In the Principles and Standards we find the following:

Part of being able to compute fluently means making smart choices about which tools to use and when. Students should have experiences that help them learn to choose among mental computation, paper-and-pencil strategies, estimation, and calculator use. The particular context, the question, and the numbers involved all play roles in those choices. Do the numbers allow a mental strategy? Does the context call for an estimate? Does the problem require repeated and tedious computations? Students should evaluate problem situations to determine whether an estimate or an exact answer is needed, using their number sense to advantage, and be able to give a rationale for their decision. (p. 36)

In addition, we find in the Focal Points the following statements with respect to estimation at the various grade levels. Notice that as the grade level advances, additional operations are included until all four operations are covered.

In the grade 2 Focal Points:
They (students) select and apply appropriate methods to estimate sums and differences or calculate them mentally, depending on the context and numbers involved. (p. 14)

In the grade 4 Focal Points:
They [students] select appropriate methods and apply them accurately to estimate products or calculate them mentally, depending on the context and numbers involved. (p. 16)

In the grade 5 Focal Points:
They [students] select appropriate methods and apply them accurately to estimate quotients or calculate them mentally, depending on the context and numbers involved. (p. 17)

In the previous sections in this chapter, we focused mainly on paper-and-pencil strategies. Next we focus on mental mathematics and computational estimation.

Mental mathematics is the process of producing an answer to a computation without using computational aids. Computational estimation is the process of forming an approximate answer to a numerical problem. Facility with estimation strategies helps to determine whether or not an answer is

Research Note
Good estimators tend to have strong self-concepts relative to mathematics, attribute their success in estimation to their ability rather than mere effort, and believe that estimation is an important tool. In contrast, poor estimators tend to have a weak self-concept relative to mathematics, attribute the success of others to effort, and believe that estimation is neither important nor useful (J. Sowder 1989).
Mental computation becomes efficient when it involves algorithms different from the standard algorithms done using pencil and paper. Also, mental computational strategies are quite personal, being dependent on a student's creativity, flexibility, and understanding of number concepts and properties. For example, consider the skills and thinking involved in computing the sum 74 + 29 by mentally representing the problem as 70 + (29 + 1) + 3 = 103 (J. Sowder 1989).

### Mental Mathematics: Addition

1. **Adding from the left**
   
   a. 67 + 36
   
   
   \[
   \begin{align*}
   60 + 30 &= 90 \\
   7 + 6 &= 13 \\
   90 + 13 &= 103
   \end{align*}
   \]
   
   (Add the tens.)(Add the units.)(Add the two sums.)

   b. 36 + 36
   
   \[
   \begin{align*}
   30 + 30 &= 60 \\
   6 + 6 &= 12 \\
   60 + 12 &= 72
   \end{align*}
   \]
   
   (Double 30.)(Double 6.)(Add the doubles.)

2. **Breaking up and bridging**
   
   \[
   \begin{align*}
   67 + 36 &= 97 \\
   67 + 30 &= 97 \\
   67 + 29 &= 96
   \end{align*}
   \]
   
   (Add the first number to the tens in the second number.)(Add this sum to the units in the second number.)

3. **Trading off**
   
   a. 67 + 3 
   
   \[
   \begin{align*}
   67 + 3 &= 70 \\
   36 - 3 &= 33 \\
   70 + 33 &= 103
   \end{align*}
   \]
   
   (Add 3 to make a multiple of 10.)(Subtract 3 to compensate for the 3 that was added.)(Add the two numbers.)

   b. 67 + 30
   
   \[
   \begin{align*}
   67 + 30 &= 97 \\
   97 - 1 &= 96
   \end{align*}
   \]
   
   (Add 30 (next multiple of 10 greater than 29.).)(Subtract 1 to compensate for the extra 1 that was added.)

4. **Using compatible numbers**
   
   Compatible numbers are numbers whose sums are easy to calculate mentally.
   
   \[
   \begin{align*}
   130 + 70 &= 200 \\
   50 + 50 &= 100 \\
   100 + 200 &= 300 \\
   300 + 20 &= 320
   \end{align*}
   \]

5. **Making compatible numbers**
   
   \[
   \begin{align*}
   25 + 75 &= 100 \\
   25 + 75 &= 100 \\
   25 + 75 &= 100 \\
   25 + 75 &= 100 \\
   25 + 75 &= 100 \\
   25 + 75 &= 100 \\
   25 + 75 &= 100 \\
   25 + 75 &= 100
   \end{align*}
   \]
   
   (25 + 75 adds to 100.)(Add 4 more units.)
Mental Mathematics: Subtraction

1. Breaking up and bridging

\[
\begin{array}{c|c}
67 & 67 - 30 = 37 \\
-36 & 37 - 6 = 31 \\
\end{array}
\]

(Subtract the tens in the second number from the first number.)

(Withdraw the units in the second number from the difference.)

2. Trading off

\[
\begin{array}{c|c}
71 & 71 + 1 = 72; 39 + 1 = 40 \\
-39 & 72 - 40 = 32 \\
\end{array}
\]

(Add 1 to both numbers. Perform the subtraction, which is easier than the original problem.)

Notice that adding 1 to both numbers does not change the answer. Why?

3. Drop the zeros

\[
\begin{array}{c|c}
8700 & 87 - 5 = 82 \\
-500 & 8200 \\
\end{array}
\]

(Notice that there are two zeros in each number. Drop these zeros and perform the computation. Then replace the two zeros to obtain proper place value.)

Another mental-mathematics technique for subtraction is called “adding up.” This method is based on the missing addend approach and is sometimes referred to as the “cashier’s algorithm.” An example of adding up or the cashier’s algorithm follows.

Example 3-5

Noah owed $11 for his groceries. He used a $50 check to pay the bill. While handing Noah the change, the cashier said, “11, 12, 13, 14, 15, 20, 30, 50.” How much change did Noah receive?

Solution  Table 3-6 shows what the cashier said and how much money Noah received each time. Since $11 plus $1 is $12, Noah must have received $1 when the cashier said $12. The same reasoning follows for $13, $14, and so on. Thus, the total amount of change that Noah received is given by $50 + $11 = $39 because $39 + $11 = $50.

Table 3-6

<table>
<thead>
<tr>
<th>What the Cashier Said</th>
<th>$11</th>
<th>$12</th>
<th>$13</th>
<th>$14</th>
<th>$15</th>
<th>$20</th>
<th>$30</th>
<th>$50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount of Money Noah Received Each Time</td>
<td>0</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$5</td>
<td>$10</td>
<td>$20</td>
</tr>
</tbody>
</table>

NOW TRY THIS 3-17 Perform each of the following computations mentally and explain what technique you used to find the answer:

a. 40 + 160 + 29 + 31
b. 3679 - 474
c. 75 + 28
d. 2500 - 700
Mental Mathematics: Multiplication

As with addition and subtraction, mental mathematics is useful for multiplication. For example, consider $8 \times 26$. Students may think of this computation in a variety of ways, as shown here.

Next we consider several of the most common strategies for performing mental mathematics using multiplication.

1. **Front-end multiplying**
   
   $64 \times 5 = 300$
   
   (Multiply the number of tens in the first number by 5.)

2. **Using compatible numbers**
   
   $2 \times 9 \times 5 \times 20 \times 5$
   
   Rearrange as $9 \times (2 \times 5) \times (20 \times 5) = 9 \times 10 \times 100 = 9000.$

3. **Thinking money**
   
   a. $64 \times 5 = 320$
   
   Think of the product as 64 nickels, which can be thought of as 32 dimes, which is cents.
   
   b. $64 \times 50 = 3200$
   
   Think of the product as 64 half-dollars, which is 32 dollars, or 3200 cents.
   
   c. $64 \times 25 = 1600$
   
   Think of the product as 64 quarters, which is 32 half-dollars, or 16 dollars. Thus we have 1600 cents.

Mental Mathematics: Division

1. **Breaking up the dividend**
   
   $7)4256$
   
   (Break up the dividend into parts.)

2. **Using compatible numbers**
   
   a. $3)105$
   
   (Look for numbers that you recognize as divisible by 3 and having a sum of 105.)

   b. $8)232$
   
   (Look for numbers that are easily divisible by 8 and whose difference is 232.)
Computational Estimation

Computational estimation may help determine whether an answer is reasonable or not. This is especially useful when the computation is done on a calculator. Some of the common estimation strategies for addition are given next.

1. **Front-end with adjustment**
   Front-end with adjustment estimation begins by focusing on the lead, or front, digits of the addition. These front, or lead, digits are added and assigned an appropriate place value. At this point we may have an underestimate that needs to be adjusted. The adjustment is made by focusing on the next group of digits. The following example shows how front-end estimation works:

   
   \[
   \begin{array}{c}
   4 + 3 + 5 \\
   12 \text{ hundred}
   \end{array}
   \]

   
   
   \[
   \begin{array}{c}
   20 \\
   100 \\
   120
   \end{array}
   \]

   
   
   
   Steps: 
   1. Add front-end digits
   2. Place value
   3. Adjust
   4. Adjusted estimate

   
   \[
   \begin{array}{c}
   4 + 3 + 5 = 12. \\
   \text{Place value} = 1200. \\
   61 + 38 \approx 100 \\
   \text{and} 
   \begin{array}{c}
   20 + 100 = 120. \\
   \text{Adjusted estimate is} \\
   1200 + 120 = 1320.
   \end{array}
   \end{array}
   \]

2. **Grouping to nice numbers**
   The strategy used to obtain the adjustment in the preceding example is the *grouping to nice numbers* strategy, which means that numbers that “nicely” fit together are grouped. Another example is given here.

   
   \[
   \begin{array}{c}
   23 \\
   39 \\
   32 \\
   64 \\
   49
   \end{array}
   \]

   
   About 100 
   Therefore, the sum is about 100 + 100, or 200.

3. **Clustering**
   *Clustering* is used when a group of numbers cluster around a common value. This strategy is limited to certain kinds of computations. In the next example, the numbers seem to cluster around 6000.

   
   \[
   \begin{array}{c}
   6200 \\
   5842 \\
   6512 \\
   5521 \\
   + 6319
   \end{array}
   \]

   
   Estimate the “average”—about 6000
   Multiply the average by the number of values to obtain 5 \cdot 6000 = 30,000.
4. **Rounding**

**Rounding** is a way of cleaning up numbers so that they are easier to handle. Rounding enables us to find approximate answers to calculations, as follows:

- \[ \begin{array}{c|c|c}
4724 & 5000 & \text{(Round 4724 to 5000.)} \\
+3192 & +3000 & \text{(Round 3192 to 3000.)} \\
\hline
8000 & & \text{(Add the rounded numbers.)}
\end{array} \]

- \[ \begin{array}{c|c|c}
1267 & 1300 & \text{(Round 1267 to 1300.)} \\
-510 & -500 & \text{(Round 510 to 500.)} \\
\hline
800 & & \text{(Subtract the rounded numbers.)}
\end{array} \]

Performing estimations requires a knowledge of place value and rounding techniques. We illustrate a rounding procedure that can be generalized to all rounding situations. For example, suppose we wish to round 4724 to the nearest thousand. We may proceed in four steps (see also Figure 3-44).

a. Determine between which two consecutive thousands the number lies.
b. Determine the midpoint between the thousands.
c. Determine which thousand the number is closer to by observing whether it is greater than or less than the midpoint. *(Not all texts use the same rule for rounding when a number falls at a midpoint.)*
d. If the number to be rounded is greater than or equal to the midpoint, round the given number to the greater thousand; otherwise, round to the lesser thousand. In this case, we round 4724 to 5000.

5. **Using the range**

It is often useful to know into what range an answer falls. The range is determined by finding a low estimate and a high estimate and reporting that the answer falls in this interval. An example follows:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Low Estimate</th>
<th>High Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>378 + 524</td>
<td>300 + 500</td>
<td>400 + 600</td>
</tr>
<tr>
<td>800</td>
<td></td>
<td>1000</td>
</tr>
</tbody>
</table>

Thus, a range for this problem is from 800 to 1000.

The student page on page 184 shows both the **rounding** and **front-end** estimation strategies applied to a problem.

### Estimation: Multiplication and Division

Examples of estimation strategies for multiplication and division are given next.

1. **Front-end**

- \[ \begin{array}{c|c|c}
524 \times 8 & 500 \times 8 = 4000 & \text{(Start multiplying at the front to obtain a first estimate.)} \\
\hline
20 \times 8 = 160 & & \text{(Multiply the next important digit by 8.)} \\
4000 + 160 = 4160 & & \text{(Adjust the first estimate by adding the two numbers.)}
\end{array} \]
ESTIMATING SUMS AND DIFFERENCES

Lesson 1-9

Key Idea
There is more than one way to estimate sums and differences.

Vocabulary
- front-end estimation
- rounding (p. 26)

Estimating Sums and Differences

How can you estimate sums?
Students at Skyline Elementary collected aluminum cans for recycling. About how many pounds of cans did they collect in all?

<table>
<thead>
<tr>
<th>Recycling Cans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade</td>
</tr>
<tr>
<td>Pounds Collected</td>
</tr>
</tbody>
</table>

You can estimate $398 + 257 + 285 + 318$ two ways.

Jon used rounding. Kylie used front-end estimation and adjusted the estimate.

I’ll round each number to the nearest hundred.

- $398 \rightarrow 400$
- $257 \rightarrow 300$
- $285 \rightarrow 300$
- $318 \rightarrow 300$

Then I’ll adjust to include the remaining numbers.

- $98 \rightarrow 100$
- $85 \rightarrow 100$
- $57 + 18 \rightarrow 100$

Less than 1,300 pounds.

About 1,300 pounds.

Source: Scott Foresman-Addison Wesley Math, Grade 5, 2008 (p. 28).
2. **Compatible numbers**

\[
\begin{array}{c|c}
5 & 4163 \\
\hline
800 & 4000 \\
\end{array}
\]

(Change 4163 to a number close to it that you know is divisible by 5.)

(Carry out the division and obtain the first estimate of 800.
Various techniques can be used to adjust the first estimate.)

**NOW TRY THIS 3-19** Estimate each of the following mentally and explain what technique you used to find the answer:

a. A sold-out concert was held in a theater with a capacity of 4525 people. Tickets were sold for $9 each. How much money was collected?

b. Fliers are to be delivered to 3625 houses and there are 42 people who will be doing the distribution. If distributed equally, how many houses will each person visit?

---

**Assessment 3-5A**

1. Compute each of the following mentally:
   a. 180 + 97 - 23 + 20 - 140 + 26
   b. 87 - 42 + 70 - 38 + 43
2. Use compatible numbers to compute each of the following mentally:
   a. 2 \cdot 9 \cdot 5 \cdot 6
   b. 8 \cdot 25 \cdot 7 \cdot 4
3. Use breaking up and bridging or front-end multiplying to compute each of the following mentally:
   a. 567 + 38
   b. 321 \cdot 3
4. Use trading off to compute each of the following mentally:
   a. 85 - 49
   b. 87 + 33
   c. 143 - 97
   d. 58 + 39
5. A car trip took 8 hr of driving at an average of 62 mph. Mentally compute the total number of miles traveled. Describe your method.
6. Compute each of the following using the *adding up* (cashier’s) algorithm:
   a. 53 - 28
   b. 63 - 47
7. Compute each of the following mentally. In each case, briefly explain your method.
   a. 86 + 37
   b. 97 + 54
   c. 230 + 60 + 70 + 44 + 40 + 6
8. Round each number to the place value indicated by the digit in bold.
   a. 5280
   b. 115,234
   c. 115,234
   d. 2,325
9. Estimate each answer by rounding.
   a. 878 ÷ 29
   b. 25,201 - 19,987
   c. 32 \cdot 28
   d. 2215 + 703 + 5967 + 975
10. Use front-end estimation with adjustment to estimate each of the following:
    a. 2215 + 3023 + 5967 + 4975
    b. 234 + 478 + 987 + 319 + 469
11. a. Would the clustering strategy of estimation be a good one to use in each of the following cases? Why or why not?
    (i) 474
    (ii) 483
    1467
    64
    + 2445
    = 503
    + 528
    b. Estimate each part of (a) using the following strategies:
    (i) Front-end
    (ii) Grouping to nice numbers
    (iii) Rounding
12. Use the range strategy to estimate each of the following. Explain how you arrived at your estimates.
    a. 22 \cdot 38
    b. 145 + 678
    c. 278 + 36
13. Suppose you had a balance of $3287 in your checking account and you wrote checks for $885, $297, $403, and $523. Estimate your balance and tell what you did and whether you think your estimate is too high or too low.
14. A theater has 38 rows with 23 seats in each row. Estimate the number of seats in the theater and tell how you arrived at your estimate.

15. Without computing, tell which of the following have the same answer. Describe your reasoning.
   a. \(44 \cdot 22\) and \(22 \cdot 11\)
   b. \(22 \cdot 32\) and \(11 \cdot 64\)
   c. \(13 \cdot 33\) and \(39 \cdot 11\)

16. The following is a list of the areas in square miles of Europe's largest countries. Mentally use this information to decide if each of the given statements is true.

<table>
<thead>
<tr>
<th>Country</th>
<th>Area (mi²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>211,207</td>
</tr>
<tr>
<td>Spain</td>
<td>194,896</td>
</tr>
<tr>
<td>Sweden</td>
<td>173,731</td>
</tr>
<tr>
<td>Finland</td>
<td>130,119</td>
</tr>
<tr>
<td>Norway</td>
<td>125,181</td>
</tr>
</tbody>
</table>

a. Sweden is less than 40,000 mi² larger than Finland.
b. France is more than twice the size of Norway.
c. France is more than 100,000 mi² larger than Norway.
d. Spain is about 21,000 mi² larger than Sweden.

17. The attendance at a World's Fair for 1 week follows:

<table>
<thead>
<tr>
<th>Day</th>
<th>Attendance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>72,250</td>
</tr>
<tr>
<td>Tuesday</td>
<td>63,891</td>
</tr>
<tr>
<td>Wednesday</td>
<td>67,490</td>
</tr>
<tr>
<td>Thursday</td>
<td>73,180</td>
</tr>
<tr>
<td>Friday</td>
<td>74,918</td>
</tr>
<tr>
<td>Saturday</td>
<td>68,480</td>
</tr>
</tbody>
</table>

Estimate the week's attendance and tell what strategy you used and why you used it.

18. In each of the following, determine if the estimate given in parentheses is high (higher than the actual answer) or low (lower than the actual answer). Justify your answers without computing the exact values.
   a. \(299 \cdot 300\) (90,000)
   b. \(6001 \div 299\) (20)
   c. \(6000 \div 299\) (20)
   d. \(999 \div 99\) (10)

19. Use your calculator to calculate \(25², 35², 45², \) and \(55²\) and then see if you can find a pattern that will let you find \(65²\) and \(75²\) mentally.

---

**Assessment 3-5B**

1. Compute each of the following mentally:
   a. \(160 + 92 - 32 + 40 - 18\)
   b. \(36 + 97 - 80 + 44\)

2. Use compatible numbers to compute each of the following mentally:
   a. \(5 \cdot 11 \cdot 3 \cdot 20\)
   b. \(82 + 37 + 18 + 13\)

3. Supply reasons for each of the first four steps given here.
   \((525 + 37) + 75 = 525 + (37 + 75)\)
   \(= 525 + (75 + 37)\)
   \(= (525 + 75) + 37\)
   \(= 600 + 37\)
   \(= 637\)

4. Use breaking and bridging or front-end multiplying to compute each of the following mentally:
   a. \(997 - 32\)
   b. \(56 \cdot 30\)

5. Use trading off to compute each of the following mentally:
   a. \(75 - 38\)
   b. \(57 + 35\)
   c. \(137 - 29\)
   d. \(78 + 49\)

6. Compute each of the following using the adding up (cashier's) algorithm:
   a. \(74 - 63\)
   b. \(73 - 57\)

7. Compute each of the following mentally. In each case, briefly explain your method.
   a. \(81 - 46\)
   b. \(98 - 19\)
   c. \(9700 - 600\)

8. Round each number to the place value indicated by the digit in bold.
   a. \(3\underline{5}87\)
   b. \(1\underline{4}8,213\)
   c. \(2\underline{3},785\)
   d. \(2,3\underline{5}7\)

9. Estimate each answer by rounding.
   a. \(937 \div 28\)
   b. \(32,285 - 18,988\)
   c. \(52 \cdot 48\)
   d. \(3215 + 3789 + 5987\)

10. Use front-end estimation with adjustment to estimate each of the following:
    a. \(2345 + 5250 + 4210 + 910\)
    b. \(345 + 518 + 655 + 270\)

11. a. Would the clustering strategy of estimation be a good one to use in each of the following cases? Why or why not?
    (i) \(318\)
    (ii) \(2350\)
    (iii) \(2314\)
    (iv) \(1987\)
    (v) \(57\)
    (vi) \(2036\)
    (vii) \(3489\)
    (viii) \(2103\)
    (ix) \(1890\)

    b. Estimate each part of (a) using the following strategies:
       (i) Front-end with adjustment
       (ii) Grouping to nice numbers
       (iii) Rounding
12. Use the range strategy to estimate each of the following. Explain how you arrived at your estimates.
   a. 32 \cdot 47
   b. 123 + 780
   c. 482 + 246

13. Tom estimated 31 \cdot 179 in the three ways shown.
   (i) 30 \cdot 200 = 6000
   (ii) 30 \cdot 180 = 5400
   (iii) 31 \cdot 200 = 6200
   Without finding the actual product, which estimate do you think is closer to the actual product? Why?

14. About 3540 calories must be burned to lose 1 lb of body weight. Estimate how many calories must be burned to lose 6 lb.

15. Without computing, tell which of the following have the same answer. Describe your reasoning.
   a. 88 \cdot 44 and 44 \cdot 22
   b. 93 \cdot 15 and 31 \cdot 45
   c. 12 \cdot 18 and 20 \cdot 17

16. In each of the following, answer the question using estimation methods if possible. If estimation is not appropriate, explain why not.
   a. Josh has $380 in his checking account. He wants to write checks for $39, $28, $59, and $250. Will he have enough money in his account to cover these checks?
   b. Gila deposited two checks into her account, one for $981 and the other for $1140. Does she have enough money in her account to cover a check for $2000 if we know she has a positive balance to start with?

17. In each of the following, determine if the estimate given in parentheses is high (higher than the actual answer) or low (lower than the actual answer). Justify your answers without computing the exact values.
   a. 398 \cdot 500 (200,000)
   b. 8001 \div 398 (20)
   c. 10,000 \div 999 (10)
   d. 1999 \div 201 (10)

18. Use your calculator to multiply several two-digit numbers times 99. Then see if you can find a pattern that will let you find the product of any two-digit number and 99 mentally.

---

Mathematical Connections 3-5

Communication
1. What is the difference between mental mathematics and computational estimation?
2. Is the front-end estimate for addition before adjustment always less than the exact sum? Explain why or why not.
3. In the new textbooks, there is an emphasis on mental mathematics and estimation. Do you think these topics are important for today's students? Why?
4. Suppose \( x \) and \( y \) are positive (greater than 0) whole numbers. If \( x \) is greater than \( y \) and you estimate \( x - y \) by rounding \( x \) up and \( y \) down, will your estimate always be too high or too low or could it be either? Explain.

Open-Ended
5. Give several examples from real-world situations where an estimate, rather than an exact answer, is sufficient.
6. a. Give a numerical example of when front-end estimation and rounding can produce the same estimate.
   b. Give an example of when they can produce a different estimate.

Cooperative Learning
7. Have each person in a group choose a different grade (3–6) textbook and make a list of mental math or estimation strategies done for each grade level. How do the lists compare?
8. As a group, and without actually finding the answers, find whether 19,876 \cdot 43 or 19,875 \cdot 44 is greater. Prepare a group response to present to the rest of the class.

Questions from the Classroom
9. Molly computed 261 − 48 by first subtracting 50 from 261 to obtain 211; then, to make up for adding 2 to 48, she subtracted 2 from 211 to obtain an answer of 209. Is her thinking correct? If not, how could you help her?
10. A student asks why he has to learn about any estimation strategy other than rounding. What is your response?
11. In order to finish her homework quickly, an elementary student does her estimation problems by using a calculator to find the exact answers and then rounds them to get her estimate. What do you tell her?
Review Problems

12. Explain why when a number is multiplied by 10 we append a zero to the number.

13. Perform each of the following divisions using both the repeated-subtraction and the standard algorithm.
   a. \(18 \div 623\)
   b. \(21 \div 493\)
   c. \(97 \div 1000\)

14. Write each of the answers in problem 13 in the form \(a = b \cdot q + r\), where \(0 \leq r < b\).

National Assessment of Educational Progress (NAEP) Questions

Which of these would be easiest to solve by using mental math?

a. \(65.12 - 28.19\)
   b. \(358 \times 2\)
   c. \(1,625 \div 3\)
   d. \(100.00 + 10.00\)

NAEP 2007, Grade 4

Mika and her mother noticed the road sign shown above while in their car on their way to Rockville. If their speed is about 65 miles per hour, approximately how many more hours are needed to finish the trip?

a. 1  b. 2  c. 3  d. 4  e. 5

NAEP 2007, Grade 8

Hint for Solving the Preliminary Problem

The use of multiple 5s, such as 55 and 555, does not often come into play and the problem can be solved using multiple 5s only twice. A major factor in solving this problem, as was shown in the example, is the use of grouping symbols.
I. Whole numbers
   A. The set of whole numbers $W$ is $\{0, 1, 2, 3, \ldots \}$.
   B. The basic operations for whole numbers are addition, subtraction, multiplication, and division.
   1. Addition: If $n(A) = a$ and $n(B) = b$, where $A \cap B = \emptyset$, then $a + b = n(A \cup B)$. The numbers $a$ and $b$ are addends and $a + b$ is the sum.
   2. Subtraction: If $a$ and $b$ are any whole numbers, then $a - b$ is the unique whole number $c$ such that $a = b + c$.
   3. Multiplication: If $a$ and $b$ are any whole numbers, and $a \neq 0$, then
      \[
      ab = \underbrace{b + b + b + \ldots + b}_{a \text{ terms}}
      \]
      where $a$ and $b$ are factors and $ab$ is the product.
   4. Multiplication: If $A$ and $B$ are sets such that $n(A) = a$ and $n(B) = b$, then $ab = n(A \times B)$.
   5. Division: If $a$ and $b$ are any whole numbers with $b \neq 0$, $a \div b$ is the unique whole number $c$ such that $bc = a$. The number $a$ is the dividend, $b$ is the divisor, and $c$ is the quotient.
   6. Division algorithm: Given any whole numbers $a$ and $b$, with $b \neq 0$, there exist unique whole numbers $q$ and $r$ such that $a = bq + r$, with $0 \leq r < b$.
   C. Properties of addition and multiplication of whole numbers
      1. Closure: If $a, b \in W$, then $a + b \in W$ and $ab \in W$.
      2. Commutative: If $a, b \in W$, then $a + b = b + a$ and $ab = ba$.
      3. Associative: If $a, b, c \in W$, then $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
      4. Identity: $0$ is the unique identity element for addition of whole numbers; $1$ is the identity element for multiplication.
      5. Distributive property of multiplication over addition: If $a, b, c \in W$, then $a(b + c) = ab + ac$.
      6. Distributive property of multiplication over subtraction: If $a, b, c \in W$, with $b \geq c$, $a(b - c) = ab - ac$.
      7. Zero multiplication property: For any whole number $a$, $a \cdot 0 = 0 = 0 \cdot a$.

D. Relations on whole numbers
   1. $a < b$ if, and only if, there is a natural number $c$ such that $a + c = b$.
   2. $a > b$ if, and only if, $b < a$.
II. Algorithms for whole-number operations
   A. Addition and subtraction algorithms
      1. Concrete models
      2. Expanded algorithms
      3. Standard algorithms
      4. Addition and subtraction with regrouping
      5. Scratch addition
      6. Equal additions
      7. Addition and subtraction in different number bases
   B. Multiplication and division algorithms
      1. Concrete models
      2. Expanded algorithms
      3. Standard algorithms
      4. Lattice multiplication
      5. Scaffolding with division
      6. Short division
      7. Multiplication and division in different number bases
III. Mental mathematics and computational estimation strategies
   A. Mental mathematics
      1. Adding from the left
      2. Breaking up and bridging
      3. Trading off
      4. Using compatible numbers
      5. Making compatible numbers
      6. Dropping the zeros
      7. Using the cashier’s algorithm (adding up)
      8. Front-end multiplying
      9. Thinking money
   10. Breaking up the dividend
   B. Computational estimation strategies
      1. Front-end
      2. Grouping to nice numbers
      3. Clustering
      4. Rounding
      5. Range
      6. Compatible numbers
1. For each of the following, identify the properties of the operation(s) for whole numbers illustrated:
   a. \(3(a + b) = 3a + 3b\)
   b. \(2 + a = a + 2\)
   c. \(16 \cdot 1 = 1 \cdot 16 = 16\)
   d. \(6(12 + 3) = 6 \cdot 12 + 6 \cdot 3\)
   e. \(3(a \cdot 2) = 3(2a)\)
   f. \(3(2a) = (3 \cdot 2)a\)

2. Using the definitions of less than or greater than given in this chapter, prove that each of the following inequalities is true:
   a. \(3 < 13\)
   b. \(12 > 9\)

3. For each of the following, find all possible replacements to make the statements true for whole numbers:
   a. \(4\Box - 37 < 27\)
   b. \(398 = \Box \cdot 37 + 28\)
   c. \(\Box \cdot (3 + 4) = \Box \cdot 3 + \Box \cdot 4\)
   d. \(42 - \Box \approx 16\)

4. Use the distributive property of multiplication over addition, other multiplication properties, and addition facts, if possible, to rename each of the following:
   a. \(3a + 7a + 5a\)
   b. \(3x^2 + 7x^2 - 5x^2\)
   c. \(x(a + b + y)\)
   d. \((x + 5)3 + (x + 5)y\)

5. How many 12-oz cans of juice would it take to give 60 people one 8-oz serving each?

6. Heidi has a brown pair and a gray pair of slacks; a brown blouse, a yellow blouse, and a white blouse; and a blue sweater and a white sweater. How many different outfits does she have if each outfit she wears consists of slacks, a blouse, and a sweater?

7. I am thinking of a whole number. If I divide it by 13, then add 89, I end up with 93. What was my original number?

8. A ski resort offers a weekend ski package for $80 per person or $6000 for a group of 80 people. Which would be the less expensive option for a group of 80?

9. Josi has a job in which she works 30 hr/wk and gets paid $5/hr. If she works more than 30 hr in a week, she receives $8/hr for each hour over 30 hr. If she worked 38 hr this week, how much did she earn?

10. In a television game show, there are five questions to answer. Each question is worth twice as much as the previous question. If the last question was worth $6400, what was the first question worth?

11. a. Think of a number.
   Add 17.
   Double the result.
   Subtract 4.
   Double the result.
   Add 20.
   Divide by 4.
   Subtract 20.
   Your answer will be your original number. Explain how this trick works.
   b. Fill in two more steps that will take you back to your original number.
   Think of a number.
   Add 18.
   Multiply by 4.
   Subtract 7.
   c. Make up a series of instructions such that you will always get back to your original number.

12. Use both the scratch and the traditional algorithms to perform the following:

   \[
   \begin{array}{c}
   \underline{316} \\
   \underline{712} \\
   \underline{+ 91}
   \end{array}
   \]

13. Use both the traditional and the lattice multiplication algorithms to perform the following:

   \[
   \begin{array}{c}
   \underline{613} \\
   \underline{\times 98}
   \end{array}
   \]

14. Use both the repeated-subtraction and the conventional algorithms to perform the following:
   a. \(912 \div 4803\)
   b. \(11 \div 1011\)
   c. \(23_{five} \div 33_{five}\)
   d. \(11_{two} \div 101_{two}\)

15. Use the division algorithm to check your answers in problem 14.

16. In some calculations a combination of mental math and a calculator is most appropriate. For example, because

   \[
   200 \cdot 97 \cdot 146 \cdot 5 = 97 \cdot 146(200 \cdot 5) = 97 \cdot 146 \times 1000
   \]

   we can calculate 97 \cdot 146 on a calculator and then mentally multiply by 1000. Show how to calculate each of the following using a combination of mental math and a calculator:
   a. \(19 \cdot 5 \cdot 194 \cdot 2\)
   b. \(379 \cdot 4 \cdot 193 \cdot 25\)
c. $8 \cdot 481 \cdot 73 \cdot 125$

d. $374 \cdot 200 \cdot 893 \cdot 50$

17. You had a balance in your checking account of $720 before writing checks for $162, $158, and $33 and making a deposit of $28. What is your new balance?

18. Jim was paid $320 a month for 6 mo and $410 a month for 6 mo. What were his total earnings for the year?

19. A soft-drink manufacturer produces 15,600 cans of his product each hour. Cans are packed 24 to a case. How many cases could be filled with the cans produced in 4 hr?

20. A limited partnership of 120 investors sold a piece of land for $461,040. If divided equally, how much did each investor receive?

21. Apples normally sell for $32¢ each. They go on sale for $3 for 69¢. How much money is saved if you purchase 2 doz apples while they are on sale?

22. The owner of a bicycle shop reported his inventory of bicycles and tricycles in an unusual way. He said he counted 126 wheels and 108 pedals. How many bikes and how many trikes did he have?

23. Perform each of the following computations:
   a. \(123_{\text{five}} + 34_{\text{five}}\)
   b. \(1010_{\text{two}} - 101_{\text{two}}\)
   c. \(23_{\text{five}} \times 34_{\text{five}}\)
   d. \(1001_{\text{two}} \times 101_{\text{two}}\)

24. Tell how to use compatible numbers mentally to perform each of the following:
   a. \(26 + 37 + 24 - 7\)
   b. \(4 \cdot 7 \cdot 9 \cdot 25\)

25. Compute each of the following mentally. Name the strategy you used to perform your mental math (strategies vary).
   a. \(63 \cdot 7\)
   b. \(85 \div 49\)
   c. \((18 \cdot 5)2\)
   d. \(2436 \div 6\)

26. Estimate the following addition using (a) front-end estimation with adjustment and (b) rounding.
   a. 543
   b. 398
   c. 255
   d. 408
   e. + 998

27. Using clustering, estimate the sum 2345 + 2854 + 2234 + 2203.

28. Explain how the standard division algorithm works for the following division:
   
   \[
   23 \quad 14) \quad 322 \\
   -28 \\
   42 \\
   -42 \\
   0
   \]

29. In some cases, the distributive property of multiplication over addition or distributive property of multiplication over subtraction can be used to obtain an answer quickly. Use one of the distributive properties to calculate each of the following in as simple a way as possible:
   a. \(999 \cdot 47 + 47\)
   b. \(43 \cdot 59 + 41 \cdot 43\)
   c. \(1003 \cdot 79 - 3 \cdot 79\)
   d. \(1001 \cdot 113 - 113\)
   e. \(101 \cdot 35\)
   f. \(98 \cdot 35\)

30. Recall that addition problems like 3478 + 521 can be written and computed using expanded notation as shown here, and answer the questions that follow.
   \[
   3 \cdot 10^3 + 4 \cdot 10^2 + 7 \cdot 10 + 8 \\
   + 5 \cdot 10^2 + 2 \cdot 10 + 1 \\
   \]
   a. Write a corresponding addition algebra problem (use \(x\) for 10) and find the answer.
   b. Write a subtraction problem and the corresponding algebra problem and find the answer.
   c. Write a multiplication problem and the corresponding algebra problem and compute the answer.


Silver, E., L. Shapiro, and A. Deutsch. “Sense Making and the Solution of Division Problems Involving Remainders: An Examination of Middle School Students’ Solution Processes and Their Interpretations.”
Selected Bibliography


