On a tour, several Egyptologists-guides were talking about the people on the latest mathematics education conference tour, all of them from either Mississippi or Tennessee. The guides could not remember the total number in the group; however, they compiled the following statistics about the group. It contained 26 Mississippi females, 17 Tennessee women, 17 Tennessee males, 29 girls, 44 Mississippi residents, 29 women, and 24 Mississippi adults. Find the total number of people in the group.
he National Council of Teachers of Mathematics (NCTM) in 2006 recognized the need for more coherence in the elementary grades mathematics curriculum. In its document *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*, the Council suggested specific topics that must be taught in grades pre–k through 8. In that document as early as pre–k, we find the following:

Children develop an understanding of the meanings of whole numbers [0, 1, 2, 3, …] and recognize the number of objects in small groups without counting—the first and most basic mathematical algorithm. They understand that number words refer to quantity. They use one-to-one correspondence to solve problems by matching sets and comparing number amounts and in counting objects to 10 and beyond. (p. 11)

In this chapter, we introduce different and early historical counting systems. Next, we present a mathematical historical development by Georg Cantor that added structure to the number system and provided methods for treating the system theoretically.

2-1 **Numeration Systems**

In this section, we introduce various number systems and compare them to the system of numbers that we use today in the United States. By comparing our current system with ancient systems that used other bases, we may develop a clearer appreciation of numbers. Our system now relies on 10 digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The written symbols for the digits, such as 2 or 5, are **numerals**. Different cultures developed different numerals over the years to represent numbers. Table 2-1 shows other representations along with how they relate to the digits 0 through 9 and the number 10.

<table>
<thead>
<tr>
<th>Table 2-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Babylonian</td>
</tr>
<tr>
<td>Egyptian</td>
</tr>
<tr>
<td>Mayan</td>
</tr>
<tr>
<td>Greek</td>
</tr>
<tr>
<td>Roman</td>
</tr>
<tr>
<td>Hindu</td>
</tr>
<tr>
<td>Arabic</td>
</tr>
<tr>
<td>Hindu-Arabic</td>
</tr>
</tbody>
</table>

Table 2-1 shows rudiments of different sets of numbers. A **numeration system** is a collection of properties and symbols agreed upon to represent numbers systematically. Through the study of various numeration systems, we explore the evolution of our current system, the Hindu-Arabic system.
Hindu-Arabic Numeration System

The Hindu-Arabic numeration system that we use today was developed by the Hindus and transported to Europe by the Arabs—hence, the name *Hindu-Arabic*. The Hindu-Arabic system relies on the following properties:

1. All numerals are constructed from the 10 digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
2. Place value is based on powers of 10, the number base of the system.

Because the Hindu-Arabic system is based on powers of 10, the system is a base-ten, or a decimal, system. **Place value** assigns a value to a digit depending on its placement in a numeral. To find the value of a digit in a whole number, we multiply the place value of the digit by its **face value**, where the face value is a digit. For example, in the numeral 5984, the 5 has place value “thousands,” the 9 has place value “hundreds,” the 8 has place value “tens,” and the 4 has place value “units,” as seen in Figure 2-1.

We could write 5984 in expanded form as $5 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10 + 4 \cdot 1$. In the expanded form of 5984, exponents are used. For example, 1000, or $10 \cdot 10 \cdot 10$, is written as $10^3$. In this case, 10 is a **factor** of the product. In general, we have the following:

**Definition of $a^n$**

If $a$ is any number and $n$ is any natural number, then

$$a^n = a \cdot a \cdot a \cdot \ldots \cdot a,$$

$n$ factors

A set of base-ten blocks, shown in Figure 2-2, consists of **units**, **longs**, **flats**, and **blocks**, representing 1, 10, 100, and 1000, respectively. Such base-ten blocks, a subset of multibase blocks, can be used to teach place value.

---

**Research Note**

Base-ten blocks improve students’ understanding of place value, accuracy in computing multidigit addition and subtraction problems, and explanations of the trading/regrouping involved. (Fuson 1992.)
Numeration Systems and Sets

1 long → 10^1 = 1 row of 10 units
1 flat → 10^2 = 1 row of 10 longs, or 100 units
1 block → 10^3 = 1 row of 10 flats, or 100 longs, or 1000 units

Students trade blocks by regrouping. That is, they take a set of base-ten blocks representing a number and trade them until they have the fewest possible pieces representing the same number. For example, suppose you have 58 units and want to trade them for other base-ten blocks. You start trading the units for as many longs as possible. Five sets of 10 units each can be traded for 5 longs. Thus, 58 units can be traded so that you now have 5 longs and 8 units. In terms of numbers, this is analogous to rewriting 58 as 5 · 10 + 8.

The use of manipulatives has been shown to improve student understanding, as seen in the Research Note on page 63.

**Example 2-1**

What is the fewest number of pieces you can receive in a fair exchange for 11 flats, 17 longs, and 16 units?

**Solution**

<table>
<thead>
<tr>
<th>11 flats</th>
<th>17 longs</th>
<th>16 units</th>
<th>(16 units = 1 long and 6 units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 long</td>
<td>6 units</td>
<td></td>
<td>(Trade)</td>
</tr>
<tr>
<td>11 flats</td>
<td>18 longs</td>
<td>6 units</td>
<td>(After the first trade)</td>
</tr>
<tr>
<td>11 flats</td>
<td>18 longs</td>
<td>6 units</td>
<td>(Trade)</td>
</tr>
<tr>
<td>12 flats</td>
<td>8 longs</td>
<td>6 units</td>
<td>(After the second trade)</td>
</tr>
<tr>
<td>12 flats</td>
<td>8 longs</td>
<td>6 units</td>
<td>(Trade)</td>
</tr>
<tr>
<td>1 block</td>
<td>2 flats</td>
<td>8 longs</td>
<td>(After the third trade)</td>
</tr>
<tr>
<td>1 block</td>
<td>2 flats</td>
<td>8 longs</td>
<td></td>
</tr>
</tbody>
</table>

Therefore, the fewest number of pieces is 1 + 2 + 8 + 6 = 17. This trading is analogous to rewriting 11 · 100 + 17 · 10 + 16 as \(1 \cdot 10^3 + 2 \cdot 10^2 + 8 \cdot 10 + 6\), which implies that there are 1286 units.

**Historical Note**

The invention of the Hindu-Arabic numeration system is considered one of the most important developments in mathematics. The system was introduced in India and then transmitted by the Arabs to North Africa and Spain and then to the rest of Europe. Historians trace the use of zero as a placeholder to the fourth century BCE (Before the Common Era). Arab mathematicians extended the decimal system to include fractions. The Italian mathematician Fibonacci, also known as Leonardo of Pisa (1170–1250), studied in Algeria and brought back with him the new numeration system, which he described and used in a book he published in 1202.
Next, we discuss other numeration systems. The study of such systems will give us a historical perspective on the development of numeration systems and will help us better understand our own system.

**Tally Numeration System**

The *tally numeration system* used single strokes, or tally marks, to represent each object that was counted; for example, the first 10 counting numbers are

\[ , ||, |||, ||||, |||||, ||||||, ||||||||, |||||||||, ||||||||||, |||||||||||| \]

In a tally system there is a correspondence between the marks and the items being counted. The system is simple, but it requires many symbols, especially when numbers become greater. Also as numbers become greater, they are harder to read.

As we see in the “Barney Google and Snuffy Smith” cartoon, the tally system can be improved by *grouping*. We see that the tallies are grouped into fives by placing a diagonal across four tallies to make a group of five. Grouping makes it easier to read the numeral.

**Egyptian Numeration System**

The Egyptian numeration system, which dates back to about 3400 BCE, used tally marks. The first nine numerals in the Egyptian system in Table 2-1 show the use of tally marks. The Egyptians improved on the system based only on tally marks by developing a *grouping system* to represent certain sets of numbers. This makes the numbers easier to record. For example, the Egyptians used a heel bone symbol, \(\cap\), to stand for a grouping of 10 tally marks.

\[ \text{\textbf{NOW TRY THIS 2-1}} \text{ Use trading with base-ten blocks (shown in Figure 2-2) to write 3 blocks, 12 flats, 11 longs, and 17 units as a Hindu-Arabic number.} \]

\[ \text{\textbf{PAW, THAT'S YORE SEVENTH DRUM-STICK!!}} \quad \text{\textbf{YEAH, BUT WHO'S COUNTIN'?}} \quad \text{\textbf{heh!! heh!!}} \]

\[ \text{\textbf{DO YA HAFTA MAKE EV'RYTHIN' EDUCATIONAL FER HIM?}} \]

\[ \text{\textbf{DORIR}} \quad \text{\textbf{JOHN}} \]

\[ \text{\textbf{©2007 by King Features Syndicate, Inc. World rights reserved.}} \]

\[ \text{\textbf{www.kingfeatures.com}} \]

\[ \text{\textbf{A Problem Solving Approach to Mathematics for Elementary School Teachers, Tenth Edition, by Rick Billstein, Shlomo Libeskind, and Johnny W. Lott. Published by Addison-Wesley. Copyright © 2010 by Pearson Education, Inc.}} \]
Table 2-2 shows other numerals that the Egyptians used in their system, and some of the symbols from the Karnak temple in Luxor are depicted in Figure 2-3.

<table>
<thead>
<tr>
<th>Egyptian Numeral</th>
<th>Description</th>
<th>Hindu-Arabic Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Vertical staff</td>
<td>1</td>
</tr>
<tr>
<td>∩</td>
<td>Heel bone</td>
<td>10</td>
</tr>
<tr>
<td>⊕</td>
<td>Scroll</td>
<td>100</td>
</tr>
<tr>
<td>≈</td>
<td>Lotus flower</td>
<td>1,000</td>
</tr>
<tr>
<td>♀</td>
<td>Pointing finger</td>
<td>10,000</td>
</tr>
<tr>
<td>⊙</td>
<td>Polliwog or burbot</td>
<td>100,000</td>
</tr>
<tr>
<td>≡</td>
<td>Astonished man</td>
<td>1,000,000</td>
</tr>
</tbody>
</table>

Note that in Figure 2-3, the symbol for 100 is carved in a different direction from that of Table 2-2.

In its simplest form, the Egyptian system involved an additive property; that is, the value of a number was the sum of the face values of the numerals. The Egyptians customarily wrote the numerals in decreasing order from left to right, as in \(\text{ⅬⅬⅬⅬⅬⅬ} \). The number can be converted to base ten as shown here:

- \(\text{ⅬⅬⅬⅬ} \) represents 100,000
- \(\text{ⅬⅬⅬ} \) represents 300 \((100 + 100 + 100)\)
- \(\text{ⅬⅬ} \) represents 20 \((10 + 10)\)
- \(\text{Ⅼ} \) represents 2 \((1 + 1)\)
- \(\text{ⅬⅬⅬⅬⅬ} \) represents 100,322
NOW TRY THIS 2-2

a. Use the Egyptian system to represent 1,312,322.
b. Use the Hindu-Arabic system to represent \( \underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{
Mayan Numeration System

In the early development of numeration systems, people frequently used parts of their bodies to count. Fingers could be matched to objects to stand for one, two, three, four, or five objects. Two hands could then stand for a set of ten objects. In warmer climates where people went barefoot, people may have used their toes as well as their fingers for counting. The Mayans introduced an attribute that was not present in the Egyptian or early Babylonian systems, namely, a symbol for zero. The Mayan system used only three symbols, which Table 2-4 shows, and based their system on 20 with vertical groupings.

<table>
<thead>
<tr>
<th>Mayan Numeral</th>
<th>Hindu-Arabic Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The symbols for the first numerals in the Mayan system are shown in Table 2-1. Notice the groupings of five, where each horizontal bar represents a group of five. Thus, the symbol for 19 was 五五五五, or three 5s and four 1s. The symbol for 20 was 六, which represents one group of 20 plus zero 1s. In Figure 2-4(a), we have 2 \cdot 5 + 3 \cdot 1, or thirteen groups of 20 plus 2 \cdot 5 + 1 \cdot 1, or eleven 1s, for a total of 271. In Figure 2-4(b), we have 3 \cdot 5 + 1 \cdot 1, or 16, groups of 20 and zero 1s, for a total of 320.

(a) (b)

Figure 2-4

In a true base-twenty system, the place value of the symbols in the third position vertically from the bottom should be 20², or 400. However, the Mayans used 20 \cdot 18, or 360, instead of 400. (The number 360 is an approximation of the length of a calendar year, which consisted of 18 months of 20 days each, plus 5 “unlucky” days.) Thus, instead of place values of 1, 20, 20², 20³, 20⁴, and so on, the Mayans used 1, 20, 20 \cdot 18, 20² \cdot 18, 20³ \cdot 18, and so on. For example, in Figure 2-5(a), we have 5 + 1 (or 6) groups of 360, plus 2 \cdot 5 + 2 (or 12) groups of 20, plus 5 + 4 (or 9) groups of 1, for a total of 2409. In Figure 2-5(b), we have 2 \cdot 5 (or 10) groups of 360, plus 0 groups of 20, plus two 1s, for a total of 3602. Spacing is important in the Mayan system. For example, if two horizontal bars are placed close together, as in 六, the symbols represent 5 + 5 = 10. If the bars are spaced apart, as in 六, then the value is 5 \cdot 20 + 5 \cdot 1 = 105.

(a) (b)

Figure 2-5
Roman Numeration System

The Roman numeration system was used in Europe in its early form from the third century BCE. It remains in use today, as seen on cornerstones, on the opening pages of books, and on the faces of some clocks. The Roman system uses only the symbols shown in Table 2-5.

<table>
<thead>
<tr>
<th>Roman Numeral</th>
<th>Hindu-Arabic Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
</tr>
<tr>
<td>V</td>
<td>5</td>
</tr>
<tr>
<td>X</td>
<td>10</td>
</tr>
<tr>
<td>L</td>
<td>50</td>
</tr>
<tr>
<td>C</td>
<td>100</td>
</tr>
<tr>
<td>D</td>
<td>500</td>
</tr>
<tr>
<td>M</td>
<td>1000</td>
</tr>
</tbody>
</table>

Roman numerals can be combined by using an additive property. For example, MDCLXVI represents $1000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666$; CCCXXXVIII represents 328, and VI represents 6.

To avoid repeating a symbol more than three times, as in IIII, a **subtractive property** was introduced in the Middle Ages. For example, I is less than V, so if it is to the left of V, it is subtracted. Thus, IV has a value of $5 - 1$, or 4, and XIX represents $100 - 10$, or 90. Some extensions of the subtractive property could lead to ambiguous results. For example, IX could be 91 or 89. By custom, 91 is written XCIX and 89 is written LXXXIX. In general, only one smaller number symbol can be to the left of a larger number symbol and the pair must be one of those listed in Table 2-6.

<table>
<thead>
<tr>
<th>Roman Numeral</th>
<th>Hindu-Arabic Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV</td>
<td>$5 - 1$, or 4</td>
</tr>
<tr>
<td>IX</td>
<td>$10 - 1$, or 9</td>
</tr>
<tr>
<td>XL</td>
<td>$50 - 10$, or 40</td>
</tr>
<tr>
<td>XC</td>
<td>$100 - 10$, or 90</td>
</tr>
<tr>
<td>CD</td>
<td>$500 - 100$, or 400</td>
</tr>
<tr>
<td>CM</td>
<td>$1000 - 100$, or 900</td>
</tr>
</tbody>
</table>

In the Middle Ages, a bar was placed over a Roman number to multiply it by 1000. The use of bars is based on a **multiplicative property**. For example, $\overline{V}$ represents $5 \cdot 1000$, or 5000, and $\overline{CDX}$ represents $410 \cdot 1000$, or 410,000. To indicate even greater numbers, more bars appear. For example, $\overline{V}$ represents $(5 \cdot 1000)1000$, or 5,000,000; $\overline{CXI}$ represents $111 \cdot 1000^3$, or 111,000,000,000; and $\overline{CXI}$ represents $110 \cdot 1000 + 1$, or 110,001.

Several properties might be used to represent some numbers, for example:

$$\overline{DCLIX} = \underbrace{(500 \cdot 1000)}_{\text{Multiplicative}} + \underbrace{(100 + 50)}_{\text{Additive}} + \underbrace{(10 - 1)}_{\text{Subtractive}} = 500,159$$

Additive
Also, the Roman system evolved over time, so there exist examples where not all rules are followed.

**Other Number Base Systems**

To better understand our base-ten system and to investigate some of the problems that students might have when learning the Hindu-Arabic system, we investigate similar systems that have different number bases.

**Base Five**

The Luo peoples of Kenya used a *quinary*, or base-five, system. A system of this type can be modeled by counting with only one hand. The digits available for counting are 0, 1, 2, 3, and 4. In the “one-hand system,” or base-five system, you count 1, 2, 3, 4, 10, where 10 represents one hand and no fingers. Counting in base five proceeds as shown in Figure 2-6. We write the small “five” below the numeral as a reminder that the number is written in base five. If no base is written, a number is assumed to be in base ten. Also note that 1, 2, 3, 4 are the same, and have the same meaning, in both base five and base ten.

<table>
<thead>
<tr>
<th>One-Hand System</th>
<th>Base-Five Symbol</th>
<th>Base-Five Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 fingers</td>
<td>0&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 0]</td>
</tr>
<tr>
<td>1 finger</td>
<td>1&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 1]</td>
</tr>
<tr>
<td>2 fingers</td>
<td>2&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 2]</td>
</tr>
<tr>
<td>3 fingers</td>
<td>3&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 3]</td>
</tr>
<tr>
<td>4 fingers</td>
<td>4&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 4]</td>
</tr>
<tr>
<td>1 hand and 0 fingers</td>
<td>10&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 10]</td>
</tr>
<tr>
<td>1 hand and 1 finger</td>
<td>11&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 11]</td>
</tr>
<tr>
<td>1 hand and 2 fingers</td>
<td>12&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 12]</td>
</tr>
<tr>
<td>1 hand and 3 fingers</td>
<td>13&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 13]</td>
</tr>
<tr>
<td>1 hand and 4 fingers</td>
<td>14&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 14]</td>
</tr>
<tr>
<td>2 hands and 0 fingers</td>
<td>20&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 20]</td>
</tr>
<tr>
<td>2 hands and 1 finger</td>
<td>21&lt;sub&gt;five&lt;/sub&gt;</td>
<td>![Base-Five Block 21]</td>
</tr>
</tbody>
</table>

Figure 2-6

Counting in base five is similar to counting in base ten. Because we have only five digits (0<sub>five</sub>, 1<sub>five</sub>, 2<sub>five</sub>, 3<sub>five</sub>, and 4<sub>five</sub>), 4<sub>five</sub> plays the role of 9 in base ten. Figure 2-7 shows how we can find the number that comes after 34<sub>five</sub> by using base-five blocks.

![Base-Five Blocks 34<sub>five</sub> + 1 = 40<sub>five</sub>](image)

Figure 2-7
What number follows $44_{\text{five}}$? There are no more two-digit numbers in the system after $44_{\text{five}}$. In base ten, the same situation occurs at 99. We use 100 to represent ten 10s, or one 100. In the base-five system, we need a symbol to represent five 5s. To continue the analogy with base ten, we use 100 to represent one group of five 5s, zero groups of five, and zero units. To distinguish from “one hundred” in base ten, the name for 100 is read “one-zero-zero base five.” The number 100 means $1 \cdot 5^2 + 0 \cdot 5^1 + 0$, whereas the number $100_{\text{five}}$ means $(1 \cdot 5^2 + 0 \cdot 5^1 + 0)_{\text{five}}$ or $1 \cdot 5^2 + 0 \cdot 5^1 + 0$, or 25.

Examples of base-five numerals along with their base-five block representations and conversions to base ten are given in Figure 2-8. Multibase blocks will be used throughout the text to illustrate various concepts.

<table>
<thead>
<tr>
<th>Base-Five Numeral</th>
<th>Base-Five Blocks</th>
<th>Base-Ten Numeral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1030_{\text{five}}$</td>
<td><img src="image" alt="Base-Five Blocks" /></td>
<td>$1 \cdot 5^3 + 0 \cdot 5^2 + 3 \cdot 5 + 0 \cdot 1 = 140$</td>
</tr>
<tr>
<td>$124_{\text{five}}$</td>
<td><img src="image" alt="Base-Five Blocks" /></td>
<td>$1 \cdot 5^2 + 2 \cdot 5 + 4 = 39$</td>
</tr>
<tr>
<td>$14_{\text{five}}$</td>
<td><img src="image" alt="Base-Five Blocks" /></td>
<td>$1 \cdot 5 + 4 = 9$</td>
</tr>
</tbody>
</table>

**Example 2-2**

Convert $11244_{\text{five}}$ to base ten.

**Solution**

$11244_{\text{five}} = 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 4 \cdot 1$

$= 1 \cdot 625 + 1 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 4 \cdot 1$

$= 625 + 125 + 50 + 20 + 4$

$= 824$

Example 2-2 suggests a method for changing a base-ten numeral to a base-five numeral using powers of 5. To convert 824 to base five, we divide by successive powers of 5 starting with the greatest power of 5 less than or equal to 824. A shorthand method for illustrating this conversion is the following:

$5^4 = 625 \rightarrow \boxed{625} \boxed{824} 1 \quad \text{How many groups of 625 in 824?}$

$5^3 = 125 \rightarrow \boxed{125} \boxed{199} 1 \quad \text{How many groups of 125 in 199?}$

$5^2 = 25 \rightarrow \boxed{25} \boxed{74} 2 \quad \text{How many groups of 25 in 74?}$

$5^1 = 5 \rightarrow \boxed{5} \boxed{24} 4 \quad \text{How many groups of 5 in 24?}$

$5^0 = 1 \rightarrow \boxed{1} \boxed{4} 4 \quad \text{How many 1s in 4?}$

Thus, $824 = 11244_{\text{five}}$. 
Calculators with the integer division feature—\(\text{INT}\) or \(\text{INTC}\) on a Texas Instruments calculator or \(\text{INT}\) or \(\text{INTC}\) on a Casio—can be used to change base-ten numbers to different number bases. For example, to convert 8 to base five, we enter and obtain \(13\)\text{five}. This implies that \(8 = 13\)\text{five}. Will this technique work to convert 34 to base five? Why or why not?

**Base Two**

Historians tell of early tribes that used base two. Some aboriginal tribes still count “one, two, two and one, two twos, two twos and one, . . .” Because base two has only two digits, it is called the **binary system**. Base two is especially important because of its use in computers. One of the two digits is represented by the presence of an electrical signal and the other by the absence of an electrical signal. Although base two works well for some purposes, it is inefficient for everyday use because multidigit numbers are reached very rapidly in counting in this system. In the following cartoon, we see an infant working with the binary system.

Conversions from base two to base ten, and vice versa, can be accomplished in a manner similar to that used for base-five conversions.

Example 2-3

a. Convert \(10111\)\text{two} to base ten.

b. Convert 27 to base two.

Solution  

a. \[10111\text{two} = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1\]  
\[= 16 + 0 + 4 + 2 + 1\]  
\[= 23\]
Another commonly used number-base system is the base-twelve, or the duodecimal (“dozens”), system. Eggs are bought by the dozen, and pencils are bought by the gross (a dozen dozen). In base twelve, there are 12 digits, just as there are 10 digits in base ten, 5 digits in base five, and 2 digits in base two. In base twelve, new symbols are needed to represent the following groups of x’s:

The new symbols chosen are T and E, respectively, so that the base-twelve digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, and E. Thus, in base twelve we count “1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, 11, 12, …, 17, 18, 19, 1T, 1E, 20, 21, 22, …, 28, 29, 2T, 2E, 30, …”

Example 2-4

a. Convert $E2T_{\text{twelve}}$ to base ten.

b. Convert 1277 to base twelve.

**Solution**

a. $E2T_{\text{twelve}} = 11 \cdot 12^2 + 2 \cdot 12^1 + 10 \cdot 1$
   
   $= 11 \cdot 144 + 24 + 10$
   
   $= 1584 + 24 + 10$
   
   $= 1618$

b. \[
\begin{array}{c|c}
144 & 1277 \\
-1152 & \\
\hline
125 & 8 \\
-120 & \\
\hline
5 & 5 \\
-5 & \\
\hline
0 &
\end{array}
\]

Thus, 1277 = 8T5_{\text{twelve}}.
Example 2-5

Rob used base twelve to write the following:

\[ g_{\text{12}} = 1050_{\text{10}} \]

What is the value of \( g \)?

**Solution** Using expanded form, we could write the following equations:

\[
\begin{align*}
g \cdot 12^2 + 3 \cdot 12 + 6 \cdot 1 &= 1050 \\
144g + 36 + 6 &= 1050 \\
144g + 42 &= 1050 \\
144g &= 1008 \\
g &= 7
\end{align*}
\]

---

**Assessment 2-1A**

1. For each of the following, tell which numeral represents the greater number and why:
   a. MCDXXIV and MCDXXXIV
   b. 4632 and 46,032
   c. \( < \bigtriangleup \) and \( \triangleleft \)
   d. \( \frac{9}{9} \) and \( \frac{9}{9} \)
   e. \( \frac{}{} \) and \( \frac{}{} \)

2. For each of the following, name both the succeeding and preceding numbers (one more and one less):
   a. MCMXLIX
   b. \( \frac{}{} \)
   c. \( \frac{}{} \)
   d. \( \frac{}{} \)

3. If the cornerstone represents when a building was built and it reads MCMXXII, when was this building built?

4. Write each of the following in Roman symbols:
   a. 121
   b. 42

5. Complete the following table, which compares symbols for numbers in different numeration systems:

<table>
<thead>
<tr>
<th>Hindu-Arabic</th>
<th>Babylonian</th>
<th>Egyptian</th>
<th>Roman</th>
<th>Mayan</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 72</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b. ( &lt; \bigtriangleup )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c. ( \frac{9}{9} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. For each of the following decimal numerals, give the place value of the underlined digit:
   a. 827,367
   b. 8,421,000

7. Rewrite each of the following as a base-ten numeral:
   a. \( 3 \cdot 10^6 + 4 \cdot 10^3 + 5 \)
   b. \( 2 \cdot 10^4 + 1 \)

8. A certain three-digit whole number has the following properties: The hundreds digit is greater than 7; the tens digit is an odd number; and the sum of the digits is 10. What could the number be?

9. Study the following counting frame. In the frame, the value of each dot is represented by the number in the box below the dot. For example, the following figure represents the number 154:

   \[
   \begin{array}{c|c|c|c|c} \text{64} & \text{8} & \text{1} \\ \hline \end{array}
   \]

   What numbers are represented in the frames in (a) and (b)?

   a. \[
   \begin{array}{c|c} \text{25} & \text{5} \\ \hline \end{array}
   \]
   b. \[
   \begin{array}{c|c|c} \text{8} & \text{4} & \text{2} \\ \hline \end{array}
   \]

10. Write the base-four numeral for the base-four representation shown.

11. Write the first 15 counting numbers for each of the following bases:
   a. base two
   b. base four

12. How many different digits are needed for base twenty?

13. Write 2032\text{four} in expanded notation.
14. Determine the greatest three-digit number in each of the following bases:
   a. base two
   b. base twelve

15. Find the number preceding and succeeding each of the following:
   a. EE0_{twelve}
   b. 100000_{two}
   c. 555_{six}

16. What, if anything, is wrong with the following numerals?
   a. 204_{four}
   b. 607_{five}

17. The smallest number of base-four blocks needed to represent 214 is ____ block(s) ____ flat(s) ____ long(s) ____ unit(s).

18. Draw multibase blocks to represent 231_{five}.

19. An introduction to base five is especially suitable for early learning in elementary school, as children can think of making change using quarters, nickels, and pennies. Use only these coins to answer the following:
   a. What is the fewest number of quarters, nickels, and pennies you can receive in a fair exchange for two quarters, nine nickels, and eight pennies?
   b. How could you use the approach in (a) to write 73 in base five?

20. Recall that with base-ten blocks, 1 long = 10 units, 1 flat = 10 longs, and 1 block = 10 flats (see Figure 2-2). In a set of multibase pieces make all possible exchanges to obtain the smallest number of pieces and write the corresponding numeral in the given base.
   a. Ten flats in base ten
   b. Twenty flats in base twelve

21. Change 42_{eight} to base two without first changing to base ten.

22. Write each of the following numbers in base ten:
   a. 432_{five}
   b. 101101_{two}
   c. 92F_{twelve}

23. You are asked to distribute $900 in prize money. The dollar amounts for the prizes are $625, $125, $25, $5, and $1. How should this $900 be distributed in order to give the fewest number of prizes?

24. Convert each of the following:
   a. 58 days to weeks and days
   b. 29 hours to days and hours

25. For each of the following, find \( b \) if possible. If not possible, tell why.
   a. \( b_{seven}^2 = 44_{ten} \)
   b. \( 5b_{twelve} = 734_{ten} \)

26. The Chinese abacus, depicted as follows, shows the number 5857. (Hint: The beads above the bar represent 5s, 50s, 500s, and 5000s.)

   Discuss how the number 5857 is depicted and show how the number 4869 could be depicted.

27. On a calculator, using only the non-zero number keys, fill the calculator's display to show the greatest number possible if each key may be used only once.

28. In a game called WIPEOUT, we are to “wipe out” digits from a calculator’s display without changing any of the other digits. “Wipeout” in this case means to replace the chosen digit(s) with a 0. For example, if the initial number is 54,321 and we are to wipe out the 4, we could subtract 4000 to obtain 50,321. Complete the following two problems and then try other numbers or challenge another person to wipe out a number from the number you have placed on the screen:
   a. Wipe out the 2s from 32,420.
   b. Wipe out the 5 from 67,357.

---

**Assessment 2-1B**

1. For each of the following, tell which numeral represents the greater number and why:
   a. MDCXXIV and MCDXXIV
   b. 3456 and 30,456
   c. \( < \langle \) and \( < \langle \) \( \rangle \)
   d. \( \langle \langle \) and \( \langle \langle \)
   e. \( \langle \langle \) and \( \langle \langle \)

2. For each of the following, name both the preceding and succeeding numbers (one more and one less):

3. Write each of the following in Roman symbols:
   a. 89
   b. 5202
4. Complete the following table, which compares symbols for numbers in different numeration systems:

<table>
<thead>
<tr>
<th>Hindu-Arabic</th>
<th>Babylonian</th>
<th>Egyptian</th>
<th>Roman</th>
<th>Mayan</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 78</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b.</td>
<td>&lt; ▼</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c.</td>
<td></td>
<td></td>
<td></td>
<td>$\varpi$</td>
</tr>
</tbody>
</table>

5. For each of the following decimal numbers, give the place value of the underlined digit:
   a. 97,998
   b. 810,485

6. Rewrite each of the following as a base-ten numeral:
   a. $3 \cdot 10^3 + 5 \cdot 10^2 + 6 \cdot 10$
   b. $9 \cdot 10^6 + 9 \cdot 10 + 9$

7. A two-digit number has the property that the units digit is 4 less than the tens digit and the tens digit is twice the units digit. What is the number?

8. On a counting frame, the following number is represented. What might the number be? Explain your reasoning.

9. Write the base-three numeral for the base-three representation shown.

10. Write the first 10 counting numbers for each of the following bases:
    a. base three
    b. base eight

11. How many different digits are needed for base eighteen?

12. Write 2022<sub>three</sub> in expanded form.

13. Determine the greatest three-digit number in each of the following bases:
    a. base three
    b. base twelve

14. Find the number preceding and succeeding each of the following:
    a. 100<sub>seven</sub>
    b. 10000<sub>two</sub>
    c. 101<sub>two</sub>

15. What, if anything, is wrong with the following numerals?
    a. $306_{\text{four}}$
    b. $1023_{\text{two}}$

16. The smallest number of base-three blocks needed to represent 79 is ______ block(s) ______ flat(s) ______ long(s) ______ unit(s).

17. Draw multibase blocks to represent 100<sub>two</sub>.

18. Using a number system based on dozen and gross, how would you describe the representation for 277?

19. Without converting to base ten, tell which is the lesser for each of the following pairs:
    a. $EET9_{12}$ or $E0T9_{12}$
    b. 1011011<sub>two</sub> or 101011<sub>two</sub>
    c. 50555<sub>six</sub> or 51000<sub>six</sub>

20. What is the smallest number of pieces of multibase blocks that can be used to write the corresponding numeral in the given base?
    a. 10 longs in base four
    b. 10 longs in base three

21. Convert each of the following base-ten numbers to a numeral in the indicated bases:
    a. 234 in base four
    b. 1876 in base twelve
    c. 303 in base three
    d. 22 in base two

22. Write each of the following numbers in base ten:
    a. $432_{\text{six}}$
    b. 11011<sub>two</sub>
    c. $E29_{12}$

23. *Who Wants the Money*, a game show, distributes prizes that are powers of 2. What is the minimum number of prizes that could be distributed from $900?

24. A coffee shop sold 1 cup, 1 pint, and 1 quart of coffee. Express the number of cups sold in base two.

25. For each of the following, find $b$, if possible. If not possible, tell why.
    a. $63_{8} = 31_{10}$
    b. $182_{12} = 1534_{6}$

26. Using only the number keys on a calculator, fill the display to show the greatest four-digit number if each key can be used only once.

---

**Mathematical Connections 2-1**

**Communication**

1. Ben claims that zero is the same as nothing. Explain how you as a teacher would respond to Ben’s statement.

2. What are the major drawbacks to each of the following systems?
   a. Egyptian
   b. Babylonian
   c. Roman
3. a. Why are large numbers in the United States written with commas separating groups of three digits?
   b. Find examples from other countries that do not use commas to separate groups of three digits.

4. Marcy bets that if you do a series of mathematical computations and activities, she can guess the color you chose. First you must find your special number by completing the following:
   Take the number of your birth month.
   Add 24.
   Add the difference you obtain when you subtract the number of your birth month from 12.
   Divide by 3.
   Add 13.
   The result is your special number.
   To each letter in the alphabet, assign a number that is the letter’s order when alphabetical, that is, \( a = 1, b = 2, c = 3, d = 4 \), and so on. Find the letter that corresponds to your special number. Next write the name of a color that starts with this letter. What color will Marcy predict is your color? Explain why this works.

Open-Ended
5. An inspector of weights and measures uses a special set of weights to check the accuracy of scales. Various weights are placed on a scale to check accuracy of any amount from 1 oz through 15 oz. What is the least number of weights the inspector needs? What weights are needed to check the accuracy of scales from 1 oz through 15 oz? From 1 oz through 31 oz?

Cooperative Learning
6. a. Create a numeration system with unique symbols and write a paragraph explaining the properties of the system.
   b. Complete the following table using the system:

<table>
<thead>
<tr>
<th>Hindu-Arabic Numeral</th>
<th>Your System Numeral</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
<tr>
<td>5,000</td>
<td></td>
</tr>
<tr>
<td>10,000</td>
<td></td>
</tr>
<tr>
<td>115,280</td>
<td></td>
</tr>
</tbody>
</table>

Mr. Harper bought 6 pints of milk. How many quarts of milk is this equal to?
   a. 3      b. 4      c. 6      d. 12

Questions from the Classroom
7. While studying different number bases, a student asks if it is possible to have a negative number for a base. What do you tell this student?
8. A student claims that the Roman system is a base-ten system since it has symbols for 10, 100, and 1000. How do you respond?
9. When using Roman numerals, a student asks whether it is correct to write \( \text{II} \), as well as \( \text{MI} \) for 1001. How do you reply?
10. A parent complains about the use of manipulatives in the classroom and likens it to the use of fingers to count. What do you say?

Third International Mathematics and Science Study (TIMSS) Questions
Which digit is in the hundreds place in 2345?
   a. 2      b. 3      c. 4      d. 5

TIMSS, 2003, Grade 4
Which of these is a name for 9740?
   a. Nine thousand seventy-four
   b. Nine thousand seven hundred forty
   c. Nine thousand seventy-four hundred
   d. Nine hundred seventy-four thousand

TIMSS, 2003, Grade 4
National Assessment of Educational Progress (NAEP) Question

1 quart = 2 pints

Mr. Harper bought 6 pints of milk. How many quarts of milk is this equal to?
   a. 3      b. 4      c. 6      d. 12

NAEP, 2007, Grade 4
Having had a look at different numeration systems and recalling some things about the Hindu-Arabic system that is used in the United States today, it is time to consider one of the major developments around the turn of the twentieth century that gave a theoretical basis for the number system we have become so accustomed to using. In the years from 1871 through 1884, Georg Cantor created set theory and had a profound effect on research and mathematics teaching.

Sets, and relations between sets, are a basis to teach children the concept of whole numbers and the concept of “less than” as well as addition, subtraction, and multiplication of whole numbers. We introduce set notation, relations between sets, set operations, and their properties. The concept of a set helps define relations and functions (see Chapter 4).

In NCTM’s Principles and Standards for School Mathematics (2000), we find:

Instructional programs from pre-kindergarten through grade 12 should enable all students to –

• understand numbers, ways of representing numbers, relationships among numbers, and number systems;

• understand meanings of operations and how they relate to one another. . . . (p. 32)

Teacher understanding of number and operation can be enhanced by a deep understanding of the mathematics behind the number system. A part of that understanding includes the notions of sets.

The Language of Sets

A set is understood to be any collection of objects. Individual objects in a set are elements, or members, of the set. For example, each letter is an element of the set of letters in the English language. The set $A$ of lowercase letters of the English alphabet can be written in set notation as follows:

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

The order in which the elements are written makes no difference, and each element is listed only once. Thus, $\{b, o, k\}$ and $\{k, o, b\}$ are considered to be the same set.

We symbolize an element belonging to a set by using the symbol $\in$. For example, $b \in A$. If an element does not belong to a set, we use the symbol $\notin$. For example, $3 \notin A$.

Georg Cantor (1845–1918) was born in St. Petersburg, Russia. His family moved to Frankfurt when he was 11. Against his father’s advice to become an engineer, Cantor pursued a career in mathematics and obtained his doctorate in Berlin at age 22. Most of his academic career was spent at the University of Halle. His hope of becoming a professor at the University of Berlin did not materialize because his work gained little recognition during his lifetime.

However, after his death Cantor’s work was praised as an “astonishing product of mathematical thought, one of the most beautiful realizations of human activity.”
For a given set to be useful in mathematics, it must be **well defined**; that is, if we are given a set and some particular object, then we must be able to tell whether the object does or does not belong to the set. For example, the set of all citizens of Pasadena, California, who ate rice on January 1, 2009, is well defined. We personally may not know if a particular resident of Pasadena ate rice or not, but that resident either belongs or does not belong to the set. On the other hand, the set of all tall people is not well defined because there is no clear meaning of “tall.”

We may use sets to define mathematical terms. For example, the set \( N \) of **natural numbers** is defined by the following:

\[
N = \{1, 2, 3, 4, \ldots \}
\]

An **ellipsis** (three dots) indicates that the sequence continues in the same manner.

Two common methods of describing sets are the **listing method** and **set-builder notation** as seen in the examples:

**Listing method:** \( C = \{1, 2, 3, 4\} \)

**Set-builder notation:** \( C = \{x \mid x \in N \text{ where } x < 5\} \)

The latter notation is read as follows:

\[
C = \{ \text{all } x \text{ such that } x \in N \text{ where } x < 5 \}
\]

Set-builder notation is useful when the individual elements of a set are not known or they are too numerous to list. For example, the set of decimals between 0 and 1 can be written as

\[
D = \{x \mid x \text{ is a decimal between 0 and 1} \}
\]

This is read “\( D \) is the set of all elements \( x \) such that \( x \) is a decimal between 0 and 1.” It would be impossible to list all the elements of \( D \). Hence the set-builder notation is indispensable here.

**Example 2-6**

Write the following sets using set-builder notation:

a. \( \{2, 4, 6, 8, 10, \ldots \} \)

b. \( \{1, 3, 5, 7, \ldots \} \)

**Solution**

a. \( \{x \mid x \text{ is an even natural number}\} \). Or because every even natural number can be written as 2 times some natural number, this set can be written as \( \{x \mid x = 2n, \text{ where } n \in N\} \) or, in a somewhat simpler form, as \( \{2n \mid n \in N\} \).

b. \( \{x \mid x \text{ is an odd natural number}\} \). Or because every odd natural number can be written as some even number minus 1, this set can be written as \( \{x \mid x = 2n - 1, \text{ where } n \in N\} \) or \( \{2n - 1 \mid n \in N\} \).
Numeration Systems and Sets

As noted earlier, the order in which the elements are listed does not matter. If sets $A$ and $B$ are equal, written then every element of $A$ is an element of $B$, and every element of $B$ is an element of $A$. If $A$ does not equal $B$, we write $A \neq B$.

Each of the following sets is described in set-builder notation. Write each of the sets by listing its elements.

a. $A = \{2k + 1 \mid k = 3, 4, 5\}$

b. $B = \{a^2 + b^2 \mid a = 2 \text{ or } 3, \text{ and } b = 2, 3, \text{ or } 4\}$

**Solution a.** We substitute $k = 3, 4, 5$ in $2k + 1$ and obtain the corresponding values shown in Table 2-7. Thus, $A = \{7, 9, 11\}$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$2k + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$2 \cdot 3 + 1 = 7$</td>
</tr>
<tr>
<td>4</td>
<td>$2 \cdot 4 + 1 = 9$</td>
</tr>
<tr>
<td>5</td>
<td>$2 \cdot 5 + 1 = 11$</td>
</tr>
</tbody>
</table>

**Table 2-7**

b. Here $a = 2 \text{ or } 3$ and $b = 2, 3, \text{ or } 4$. Table 2-8 shows all possible combinations of $a$ and $b$ and the corresponding values of $a^2 + b^2$. Thus, $B = \{8, 13, 20, 18, 25\}$. Notice that 13 appears twice in the table but only once in the set. Why?

<table>
<thead>
<tr>
<th>$a$</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2^2 + 2^2 = 8$</td>
<td>$2^2 + 3^2 = 13$</td>
<td>$2^2 + 4^2 = 20$</td>
</tr>
<tr>
<td>3</td>
<td>$3^2 + 2^2 = 13$</td>
<td>$3^2 + 3^2 = 18$</td>
<td>$3^2 + 4^2 = 25$</td>
</tr>
</tbody>
</table>

**Table 2-8**

As noted earlier, the order in which the elements are listed does not matter. If sets $A$ and $B$ are equal, written $A = B$, then every element of $A$ is an element of $B$, and every element of $B$ is an element of $A$. If $A$ does not equal $B$, we write $A \neq B$.

**Definition of Equal Sets**

Two sets are **equal** if, and only if, they contain exactly the same elements.

**One-to-One Correspondence**

One of the most useful tools in mathematics is a **one-to-one correspondence** between two sets. Here the sets may be equal or not. For example, consider the set of people $P = \{\text{Tomas, Dick, Mari}\}$ and the set of swimming lanes $S = \{1, 2, 3\}$. Suppose each person in $P$ is to swim in a lane numbered 1, 2, or 3 so that no two people swim in the same lane. Such a person-lane pairing is a one-to-one correspondence. One way to exhibit a one-to-one correspondence is shown in Figure 2-9 with arrows connecting corresponding elements.
Other possible one-to-one correspondences exist between the sets $P$ and $S$. All six possible one-to-one correspondences between sets $P$ and $S$ can be listed as follows:

1. Tomas $\leftrightarrow 1$
   Dick $\leftrightarrow 2$
   Mari $\leftrightarrow 3$

2. Tomas $\leftrightarrow 1$
   Dick $\leftrightarrow 3$
   Mari $\leftrightarrow 2$

3. Tomas $\leftrightarrow 2$
   Dick $\leftrightarrow 1$
   Mari $\leftrightarrow 3$

4. Tomas $\leftrightarrow 2$
   Dick $\leftrightarrow 3$
   Mari $\leftrightarrow 1$

5. Tomas $\leftrightarrow 3$
   Dick $\leftrightarrow 1$
   Mari $\leftrightarrow 2$

6. Tomas $\leftrightarrow 3$
   Dick $\leftrightarrow 2$
   Mari $\leftrightarrow 1$

Notice that the listing in (1) as well as Figure 2-9 represent a single one-to-one correspondence between the sets $P$ and $S$. The correspondence Tomas $\leftrightarrow 1$ can also be a one-to-one correspondence but between two different sets, namely, the sets \{Tomas\} and \{1\}. The complete set of one-to-one correspondences between sets $P$ and $S$ is seen in Table 2-9.

<table>
<thead>
<tr>
<th>Pairings</th>
<th>Lanes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1. Tomas</td>
<td>Dick</td>
</tr>
<tr>
<td>2. Tomas</td>
<td>Mari</td>
</tr>
<tr>
<td>3. Dick</td>
<td>Tomas</td>
</tr>
<tr>
<td>4. Dick</td>
<td>Mari</td>
</tr>
<tr>
<td>5. Mari</td>
<td>Tomas</td>
</tr>
<tr>
<td>6. Mari</td>
<td>Dick</td>
</tr>
</tbody>
</table>

The general definition of one-to-one correspondence follows:

**Definition of One-to-One Correspondence**

If the elements of sets $P$ and $S$ can be paired so that for each element of $P$ there is exactly one element of $S$ and for each element of $S$ there is exactly one element of $P$, then the two sets $P$ and $S$ are said to be in one-to-one correspondence.

**NOW TRY THIS 2-5** Consider a set of four people \{A, B, C, D\} and a set of four swimming lanes \{1, 2, 3, 4\}.

a. Exhibit all the one-to-one correspondences between the two sets.
b. How many such one-to-one correspondences are there?c. Find the number of one-to-one correspondences between two sets with five elements each and explain your reasoning.

A tree diagram also lists the possible one-to-one correspondences in Figure 2-10. To read the tree diagram and see the one-to-one correspondence, we follow each branch. The
Observe in Figure 2-10 when assigning a swimmer to lane 1 we have a choice of three people: Tomas, Dick, or Mari. If we put Tomas in lane 1, then he cannot be in lane 2, and hence the second lane must be occupied by either Dick or Mari. In the same way, we see that if Dick is in lane 1, then there are two choices for lane 2: Tomas or Mari. Similarly, if Mari is in lane 1, then again there are two choices for the second lane: Tomas or Dick. Thus, for each of the three ways we can fill the first lane, there are two subsequent ways to fill the second lane, and hence there are \(2 \times 2 = 4\) or \(3 \times 2\) or \(6\) ways to arrange the swimmers in the first two lanes. Notice that for each arrangement of the swimmers in the first two lanes, there remains only one possible swimmer to fill the third lane. For example, if Mari fills the first lane and Dick fills the second, then Tomas must be in the third. Thus, the total number of arrangements for the three swimmers is equal to \(3 \times 2\), or 6.

Similar reasoning can be used to find how many ice-cream arrangements are possible on a two-scoop cone if 10 flavors are offered. If we count chocolate and vanilla (chocolate on bottom and vanilla on top) different from vanilla and chocolate (vanilla on bottom and chocolate on top) and allow two scoops to be of the same flavor, we can proceed as follows. There are 10 choices for the first scoop, and for each of these 10 choices there are 10 subsequent choices for the second scoop. Thus, the total number of arrangements is \(10 \times 10\) or \(100\).

The counting argument used to find the number of possible one-to-one correspondences between the set of swimmers and the set of lanes and the previous problem about ice-cream-scoop arrangements are examples of the Fundamental Counting Principle.

**Theorem 2–1: Fundamental Counting Principle**

If event \(M\) can occur in \(m\) ways and, after it has occurred, event \(N\) can occur in \(n\) ways, then event \(M\) followed by event \(N\) can occur in \(mn\) ways.
Equivalent Sets

Closely associated with one-to-one correspondences is the concept of **equivalent sets**. For example, suppose a room contains 20 chairs and one student is sitting in each chair with no one standing. There is a one-to-one correspondence between the set of chairs and the set of students in the room. In this case, the set of chairs and the set of students are equivalent sets.

**Definition of Equivalent Sets**

Two sets $A$ and $B$ are **equivalent**, written $A \sim B$, if and only if there exists a one-to-one correspondence between the sets.

The term *equivalent* should not be confused with *equal*. The difference should be made clear by Example 2-8.

**Example 2-8**

Let

$A = \{p, q, r, s\}, \quad B = \{a, b, c\}, \quad C = \{x, y, z\}, \quad \text{and} \quad D = \{b, a, c\}.$

Compare the sets, using the terms *equal* and *equivalent*.

**Solution**

Each set is both equivalent to and equal to itself.

Sets $A$ and $B$ are not equivalent ($A \not\sim B$) and not equal ($A \neq B$).

Sets $A$ and $C$ are not equivalent ($A \not\sim C$) and not equal ($A \neq C$).

Sets $A$ and $D$ are not equivalent ($A \not\sim D$) and not equal ($A \neq D$).

Sets $B$ and $C$ are equivalent ($B \sim C$) but not equal ($B \neq C$).

Sets $B$ and $D$ are equivalent ($B \sim D$) and equal ($B = D$).

Sets $C$ and $D$ are equivalent ($C \sim D$) but not equal ($C \neq D$).

**NOW TRY THIS 2-7**

a. If two sets are equivalent, are they necessarily equal? Explain why or why not.

b. If two sets are equal, are they necessarily equivalent? Explain why or why not.

Cardinal Numbers

The concept of one-to-one correspondence can be used to consider the notion of two sets having the same number of elements. Without knowing how to count, a child might tell that there are as many fingers on the left hand as on the right hand by matching the fingers on one hand with the fingers on the other hand, as in Figure 2-11. Naturally placing the
fingers so that the left thumb touches the right thumb, the left index finger touches the right index finger, and so on, exhibits a one-to-one correspondence between the fingers of the two hands. Similarly, without counting, children realize that if every student in a class sits in a chair and no chairs are empty, there are as many chairs as students, and vice versa.

Figure 2-11

One-to-one correspondence between sets helps explain the concept of a number. Consider the five sets \{a, b\}, \{p, q\}, \{x, y\}, \{b, a\}, and \{* , #\}; the sets are equivalent to one another and share the property of “twoness”; that is, these sets have the same cardinal number, namely, 2. The **cardinal number** of a set \(S\), denoted \(n(S)\), indicates the number of elements in the set \(S\). If \(S = \{a, b\}\), the cardinal number of \(S\) is 2, and we write \(n(S) = 2\). If two sets, \(A\) and \(B\), are equivalent, then \(A\) and \(B\) have the same cardinal number; that is, \(n(A) = n(B)\).

A set that contains no elements has cardinal number 0 and is an **empty**, or **null**, set. The empty set is designated by the symbol \(\emptyset\) or \(\{ \}\). Two examples of sets with no elements are the following:

\[ C = \{x \mid x \text{ was a state of the United States before 1200 CE}\} \]
\[ D = \{x \mid x \text{ is a natural number less than 1}\} \]

**REMARK** The empty set is often incorrectly recorded as \(\{\emptyset\}\). This set is not empty but contains one element. Likewise, \(\{0\}\) does not represent the empty set. Why?

A set is a **finite set** if the cardinal number of the set is zero or a natural number. The set of natural numbers \(N\) is an **infinite set**; it is not finite. The set \(W\), containing all the natural numbers and 0, is the set of **whole numbers**: \(W = \{0, 1, 2, 3, \ldots\}\). \(W\) is an infinite set.

The following “Peanuts” cartoon demonstrates some set theory concepts related to addition, though a child would not be expected to know all these concepts to add 2 and 2.
More About Sets

The universal set, or the universe, denoted $U$, is the set that contains all elements being considered in a given discussion. Suppose $U = \{x \mid x$ is a person living in California$\}$ and $F = \{x \mid x$ is a female living in California$\}$. The universal set, $U$, and set $F$ can be represented by a diagram, as in Figure 2-12(a). The universal set is represented by a large rectangle, and $F$ is indicated by the circle inside the rectangle, as shown in Figure 2-12(a). This figure is an example of a Venn diagram, named after the Englishman John Venn (1834–1923), who used such diagrams to illustrate ideas in logic. The set of elements in the universe that are not in $F$, denoted by $\overline{F}$, is the set of males living in California and is the complement of $F$. It is represented by the shaded region in Figure 2-12(b).

**Definition of Set Complement**

The complement of a set $F$, written $\overline{F}$, is the set of all elements in the universal set $U$ that are not in $F$; that is, $\overline{F} = \{x \mid x \in U$ and $x \notin F\}$. 

[Figure 2-12]
Example 2-9

a. If \( U = \{a, b, c, d\} \) and \( B = \{c, d\} \), find (i) \( B \); (ii) \( U \); (iii) \( \emptyset \).

b. If \( U = \{x \mid x \text{ is an animal in the zoo}\} \) and \( S = \{x \mid x \text{ is a snake in the zoo}\} \), describe \( S \).

c. If \( U = N, E = \{2, 4, 6, 8, \ldots\} \), and \( O = \{1, 3, 5, 7, \ldots\} \), find (i) \( E \); (ii) \( \overline{O} \).

Solution

a. (i) \( E = \{a, b\} \); (ii) \( U = \emptyset \); (iii) \( \emptyset = U \)

b. Because the individual animals in the zoo are not known, \( S \) must be described using set-builder notation:

\[ S = \{x \mid x \text{ is a zoo animal that is not a snake}\} \]

c. (i) \( E = O \); (ii) \( \overline{O} = E \)

Subsets

Consider the sets \( A = \{1, 2, 3, 4, 5, 6\} \) and \( B = \{2, 4, 6\} \). All the elements of \( B \) are contained in \( A \) and we say that \( B \) is a subset of \( A \). We write \( B \subseteq A \). In general, we have the following:

**Definition of Subset**

\( B \) is a subset of \( A \), written \( B \subseteq A \), if, and only if, every element of \( B \) is an element of \( A \).

This definition allows \( B \) to be equal to \( A \). The definition is written with the phrase “if, and only if,” which means “if \( B \) is a subset of \( A \), then every element of \( B \) is an element of \( A \), and if every element of \( B \) is an element of \( A \), then \( B \) is a subset of \( A \).” If both \( A \subseteq B \) and \( B \subseteq A \), then \( A = B \).

When a set \( A \) is not a subset of another set \( B \), we write \( A \not\subseteq B \). To show that \( A \not\subseteq B \), we must find at least one element of \( A \) that is not in \( B \). If \( A = \{1, 3, 5\} \) and \( B = \{1, 2, 3\} \), then \( A \) is not a subset of \( B \) because 5 is an element of \( A \) but not of \( B \). Likewise, \( B \not\subseteq A \) because 2 belongs to \( B \) but not to \( A \).

It is not obvious how the empty set fits the definition of a subset because no elements in the empty set are elements of another set. To investigate this problem, we use the strategies of indirect reasoning and looking at a special case.

For the set \( \{1, 2\} \), either \( \emptyset \subseteq \{1, 2\} \) or \( \emptyset \not\subseteq \{1, 2\} \). Suppose \( \emptyset \not\subseteq \{1, 2\} \). Then there must be some element in \( \emptyset \) that is not in \( \{1, 2\} \). Because the empty set has no elements, there cannot be an element in the empty set that is not in \( \{1, 2\} \). Consequently, \( \emptyset \not\subseteq \{1, 2\} \) is false. Therefore, the only other possibility, \( \emptyset \subseteq \{1, 2\} \), is true. The same reasoning can be applied in the case of the empty set and any other set.

If \( B \) is a subset of \( A \) and \( B \) is not equal to \( A \), then \( B \) is a proper subset of \( A \), written \( B \subset A \). This means that every element of \( B \) is contained in \( A \) and there is at least one element of \( A \) that is not in \( B \). To indicate a proper subset, sometimes a Venn diagram like the one shown in Figure 2-13 is used, showing a dot (an element) in \( A \) that is not in \( B \).

![Figure 2-13](image-url)
Subsets and elements of sets are often confused. We say that $2$ is not a set, we cannot substitute the symbol for $\in$ between $2$ and $\{1, 2, 3\}$ in a true sentence.

Given $A = \{1, 2, 3, 4, 5\}, B = \{1, 3\}, P = \{x \mid x = 2^n - 1, \text{ where } n \in N\}$:

a. Which sets are subsets of each other?

b. Which sets are proper subsets of each other?

c. If $C = \{2k \mid k \in N\}$ and $D = \{4k \mid k \in N\}$, show that one of the sets is a subset of the other.

Solution

a. Because $2^1 - 1 = 1, 2^2 - 1 = 3, 2^3 - 1 = 7, 2^4 - 1 = 15, 2^5 - 1 = 31$, and so on, $P = \{1, 3, 7, 15, 31, \ldots \}$.

Thus, $B \subseteq P$. Also $B \subseteq A, A \subseteq A, B \subseteq B$ and $P \subseteq P$.

b. $B \subseteq A$ and $B \subseteq P$

c. Because $4k = 2(2k)$, every element of $D$ is an element of $C$. Thus, $D \subseteq C$.

NOW TRY THIS 2-9

a. Suppose $A \subseteq B$. Can we always conclude that $A \subseteq B$?

b. If $A \subseteq B$, does it follow that $A \subseteq B$?

Subsets and elements of sets are often confused. We say that $2 \in \{1, 2, 3\}$. But because $2$ is not a set, we cannot substitute the symbol $\in$ for $\subseteq$. However, $\{2\} \subseteq \{1, 2, 3\}$ and $\{2\} \subseteq \{1, 2, 3\}$. Notice that the symbol $\in$ cannot be used between $\{2\}$ and $\{1, 2, 3\}$ in a true sentence.

NOW TRY THIS 2-10

Convince a classmate that the following are true:

a. The empty set is a subset of itself.

b. The empty set is not a proper subset of itself.

Inequalities: An Application of Set Concepts

The notion of a proper subset and the concept of one-to-one correspondence can be used to define the concept of “less than” among natural numbers. The set $\{a, b, c\}$ has fewer elements than the set $\{w, x, y, z\}$ because when we try to pair the elements of the two sets, as in

\[
\begin{align*}
\{a, b, c\} & \quad \uparrow \quad \uparrow \quad \uparrow \\
\{x, y, z, w\} &
\end{align*}
\]

we see that there is an element of the second set that is not paired with an element of the first set. The set $\{a, b, c\}$ is equivalent to a proper subset of the set $\{x, y, z, w\}$.

In general, if $A$ and $B$ are finite sets, $A$ has fewer elements than $B$ if $A$ is equivalent to a proper subset of $B$. We say that $n(A)$ is less than $n(B)$ and write $n(A) < n(B)$. We say that $a$ is greater than $b$, written $a > b$, if, and only if, $b < a$. Defining the concept of “less than or equal to” in a similar way is explored in Assessment 2-2A and 2-2B.

We have just seen that if $A$ and $B$ are finite sets and $A \subseteq B$, then $A$ has fewer elements than $B$ and it is not possible to find a one-to-one correspondence between the sets. Consequently, $A$ and $B$ are not equivalent. However, when both sets are infinite and $A \subseteq B$, the sets could be equivalent. For example, consider the set $N$ of counting numbers and the set
Research Note

Tsamir and Triosh (1999) reported that with infinite sets, students frequently wound up with contradictory conclusions and did not use one-to-one correspondences. These results were not inconsistent with the thinking in mathematics before Cantor’s work.

Problem Solving

Passing a Senate Measure

A committee of senators consists of Abel, Baro, Carni, and Davis. Suppose each member of the committee has one vote and a simple majority is needed to either pass or reject any measure. A measure that is neither passed nor rejected is considered to be blocked and will be voted on again. Determine the number of ways a measure could be passed or rejected and the number of ways a measure could be blocked.

Understanding the Problem

We are asked to determine how many ways the committee of four could pass or reject a proposal and how many ways the committee of four could block a proposal. To pass or reject a proposal requires a winning coalition, that is, a group of senators who can pass or reject the proposal, regardless of what the others do. To block a proposal, there must be a blocking coalition, that is, a group who can prevent any proposal from passing but who cannot reject the measure.

Devising a Plan

To solve the problem, we can make a list of subsets of the set of senators. Any subset of the set of senators with three or four members will form a winning coalition. Any subset of the set of senators with exactly two members will form a blocking coalition.

Carrying Out the Plan

We list all subsets of the set \( S = \{ \text{Abel, Baro, Carni, Davis} \} \) that have at least three elements and all subsets that have exactly two elements. For ease, we identify the members as follows: \( A \) — Abel, \( B \) — Baro, \( C \) — Carni, \( D \) — Davis. All the subsets are given next:

\[
\begin{align*}
\emptyset & \quad \{A\} \quad \{A, B\} \quad \{A, B, C\} \quad \{A, B, C, D\} \\
\{B\} & \quad \{A, C\} \quad \{A, B, D\} \\
\{C\} & \quad \{A, D\} \quad \{A, C, D\} \\
\{D\} & \quad \{B, C\} \quad \{B, C, D\} \\
& \quad \{B, D\} \\
& \quad \{C, D\}
\end{align*}
\]
There are five subsets with at least three members that can form a winning coalition to pass or reject a measure and six subsets with exactly two members that can block a measure.

Looking Back  Other questions that might be considered include the following:

1. How many minimal winning coalitions are there? In other words, how many subsets are there of which no proper subset could pass a measure?
2. Devise a method to solve this problem without listing all subsets.
3. In “Carrying Out the Plan,” 16 subsets of \{A, B, C, D\} are listed. Use that result to systematically list all the subsets of a committee of five senators. Can you find the number of subsets of the 5-member committee without actually counting the subsets?

NOW TRY THIS 2-11 Suppose a committee of U.S. senators consists of five members.

a. Compare the number of winning coalitions having exactly four members with the number of senators on the committee. What is the reason for the result?

b. Compare the number of winning coalitions having exactly three members with the number of subsets of the committee having exactly two members. What is the reason for the result?

Number of Subsets of a Set

How many subsets can be made from a set containing \(n\) elements? To obtain a general formula, we use the strategy of trying simpler cases first.

1. If \(P = \{a\}\), then \(P\) has two subsets, \(\emptyset\) and \(\{a\}\).
2. If \(Q = \{a, b\}\), then \(Q\) has four subsets, \(\emptyset, \{a\}, \{b\}, \text{ and } \{a, b\}\).
3. If \(R = \{a, b, c\}\), then \(R\) has eight subsets, \(\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \text{ and } \{a, b, c\}\).

An alternative strategy for listing the number of subsets of a given set is to use a tree diagram. For example, tree diagrams for the subsets of \(Q = \{a, b\}\) and \(R = \{a, b, c\}\) are given in Figure 2-14.

![Tree Diagrams for Subsets](image)

(a) Subsets of \(Q = \{a, b\}\)

(b) Subsets of \(R = \{a, b, c\}\)

Figure 2-14

Using the information from these cases, we make a table and search for a pattern, as in Table 2-10.
Table 2-10

<table>
<thead>
<tr>
<th>Number of Elements</th>
<th>Number of Subsets</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2, or $2^1$</td>
</tr>
<tr>
<td>2</td>
<td>4, or $2^2$</td>
</tr>
<tr>
<td>3</td>
<td>8, or $2^3$</td>
</tr>
</tbody>
</table>

Table 2-10 suggests that for four elements, there might be $2^4$, or 16, subsets. Is this correct? If $S = \{a, b, c, d\}$, then all the subsets of $R = \{a, b, c\}$ are also subsets of $S$. Eight new subsets are also formed by including the element $d$ in each of the eight subsets of $R$. Thus, there are twice as many subsets of set $S$ (with four elements) as there are of set $R$ (with three elements). Consequently, there are $2 \cdot 8$, or $2^4$, subsets of a set with four elements. Because including one more element in a finite set doubles the number of possible subsets of the new set, a set with five elements will have $2 \cdot 2^4$, or $2^5$, subsets and so on. In each case, the number of elements and the power of 2 used to obtain the number of subsets are equal. Therefore, if there are $n$ elements in a set, $2^n$ subsets can be formed. If we apply this result to the empty set—that is when $n = 0$—then we have $2^0 = 1$. The pattern is meaningful because the empty set has only one subset—itself.

NOW TRY THIS 2-12

a. How many proper subsets does a set with four elements have?  
b. How many proper subsets does a set with $n$ elements have?

Assessment 2-2A

1. Write the following sets using the listing method or using set-builder notation:  
   a. The set of letters in the word mathematics  
   b. The set of natural numbers greater than 20
2. Rewrite the following using mathematical symbols:  
   a. $P$ is equal to the set containing $a$, $b$, $c$, and $d$.  
   b. The set consisting of the elements 1 and 2 is a proper subset of $\{1, 2, 3, 4\}$.  
   c. The set consisting of the elements 0 and 1 is not a subset of $\{1, 2, 3, 4\}$.  
   d. 0 is not an element of the empty set.
3. Which of the following pairs of sets can be placed in one-to-one correspondence?  
   a. $\{1, 2, 3, 4, 5\}$ and $\{m, n, o, p, q\}$  
   b. $\{a, b, c, d, e, f, \ldots, m\}$ and $\{1, 2, 3, 4, 5, 6, \ldots, 13\}$  
   c. $\{x \mid x$ is a letter in the word mathematics$\}$ and $\{1, 2, 3, 4, \ldots, 11\}$
4. How many one-to-one correspondences are there between two sets with  
   a. 6 elements each?  
   b. $n$ elements each?
5. How many one-to-one correspondences are there between the sets $\{x, y, z, u, v\}$ and $\{1, 2, 3, 4, 5\}$ if in each correspondence  
   a. $x$ must correspond to 5?  
   b. $x$ must correspond to 5 and $y$ to 1?  
   c. $x$, $y$, and $z$ must correspond to odd numbers.
6. Which of the following represent equal sets?
   \[ A = \{a, b, c, d\} \quad B = \{x, y, z, w\} \]
   \[ C = \{c, d, a, b\} \quad D = \{x \mid 1 \leq x \leq 4 \text{ where } x \in N\} \]
   \[ E = \emptyset \quad F = \{\emptyset\} \]
   \[ G = \{0\} \quad H = \{\} \]
   \[ I = \{x \mid x = 2n+1 \text{ where } n \in W\} \]
   \[ J = \{x \mid x = 2n - 1 \text{ where } n \in N\} \]

7. Find the cardinal number of each of the following sets:
   a. {101, 102, 103, \ldots, 1100}
   b. {1, 3, 5, \ldots, 1001}
   c. {1, 2, 4, 8, 16, \ldots, 1024}
   d. {x \mid x = k^2 \text{ where } k = 1, 2, 3, \ldots, 100}
   e. \{i + j \mid i \in \{1, 2, 3\} \text{ and } j \in \{1, 2, 3\}\}

8. If \( U \) is the set of all college students and \( A \) is the set of all college students with a straight-A average, describe \( A \).

9. Suppose \( B \) is a proper subset of \( C \).
   a. If \( n(C) = 8 \), what is the maximum number of elements in \( B \)?
   b. What is the least possible number of elements in \( B \)?

10. Suppose \( C \) is a subset of \( D \) and \( D \) is a subset of \( C \).
    a. If \( n(C) = 5 \), find \( n(D) \).
    b. What other relationship exists between sets \( C \) and \( D \)?

11. Indicate which symbol, \( \subseteq \) or \( \subset \), makes each of the following statements true:
    a. 0 ______ \( \emptyset \)
    b. \{1\} ______ \{1, 2\}
    c. 1024 ______ \{x \mid x = 2^n \text{ where } n \in N\}
    d. 3002 ______ \{x \mid x = 3n - 1 \text{ where } n \in N\}

12. Indicate which symbol, \( \subseteq \) or \( \subset \), makes each part of problem 11 true.

13. Answer each of the following. If your answer is no, tell why.
    a. If \( A = B \), can we always conclude that \( A \subseteq B \)?
    b. If \( A \subseteq B \), can we always conclude that \( A \subset B \)?
    c. If \( A \subset B \), can we always conclude that \( A \subseteq B \)?
    d. If \( A \subseteq B \), can we always conclude that \( A = B \)?

14. Use the definition of \( \text{less than} \) to show each of the following:
    a. 3 < 100
    b. 0 < 3

15. On a certain senate committee there are seven senators: Abel, Brooke, Cox, Dean, Eggers, Funk, and Gage. Three of these members are to be appointed to a subcommittee. How many possible subcommittees are there?

16. How many two-digit numbers in base ten can be formed if the tens digit cannot be 0 and no digit can be repeated?

---

### Section 2-2 Describing Sets

**Assessment 2-2B**

1. Write the following sets using the listing method or set-builder notation:
   a. the set of letters in the word \( \text{geometry} \)
   b. the set of natural numbers greater than 7

2. Rewrite the following using mathematical symbols:
   a. \( Q \) is equal to the set whose elements are \( a, b, \) and \( c \).
   b. The set containing 1 and 3 only is a proper subset of the set of natural numbers.
   c. The set containing 1 and 3 only is not a subset of \( \{1, 4, 6, 7\} \).
   d. The empty set does not contain 0 as an element.

3. Which of the following pairs of sets can be placed in a one-to-one correspondence?
   a. \{1, 2, 3, 4\} and \{\{w, c, y, z\}\}
   b. \{1, 2, 3, \ldots, 25\} and \{a, b, c, d, \ldots, x, y\}
   c. \{x \mid x \text{ is a letter in the word } \text{geometry}\} \text{ and } \{1, 2, 3, 4, 5, 6, 7, 8\}

4. How many one-to-one correspondences exist between two sets with
   a. 8 elements each?
   b. \( n - 1 \) elements each?

5. How many one-to-one correspondences are there between the sets \( \{a, b, c, d\} \) and \{1, 2, 3, 4\} if in each correspondence
   a. \( b \) must correspond to 3?
   b. \( b \) must correspond to 3 and \( d \) to 4?
   c. \( a \) and \( c \) must correspond to even numbers?

6. Which of the following represent unequal sets?
   a. \( A = \{a, b, c, d\} \quad B = \{x, y, z, w\} \)
   b. \( C = \{c, d, a, b\} \quad D = \{x \mid 1 \leq x \leq 4 \text{ where } x \in N\} \)
   c. \( E = \emptyset \quad F = \{\emptyset\} \)
   d. \( G = \{0\} \quad H = \{\} \)
   e. \( I = \{x \mid x = 2n + 1 \text{ where } n \in W\} \)
   f. \( J = \{x \mid x = 2n - 1 \text{ where } n \in N\} \)

7. Find the cardinal number of each of the following sets:
   a. \{9, 10, 11, \ldots, 99\}
   b. \{2, 4, 6, 8, \ldots, 2002\}
   c. \{0, 1, 3, 7, 15, \ldots, 1023\}
   d. \{x^2 \mid x = 1, 3, 5, 7, \ldots, 99\}
   e. \{i+j \mid i \in \{1, 2, 3\} \text{ and } j \in \{1, 2, 3\}\}
92  Numeration Systems and Sets

8. If \( U \) is the set of all women and \( G \) is the set of alumnae of Georgia State University, describe \( G \).

9. Suppose \( A \subseteq B \).
   a. What is the minimum number of elements in set \( A \)?
   b. Is it possible for set \( B \) to be the empty set? If so, give an example of sets \( A \) and \( B \) satisfying this. If not, explain why not.

10. If two sets are subsets of each other, what other relationships must they have?

11. Indicate which symbol, \( \in \) or \( \not\in \), makes each of the following statements true:
   a. \( \emptyset \) \( \subseteq \) \( \emptyset \)
   b. \( \{2\} \) \( \not\subseteq \) \( \{3, 2, 1\} \)
   c. \( 1022 \) \( \subseteq \) \( \{s | s = 2^n - 2 \text{ where } n \text{ is an element of } N\} \)
   d. \( 3004 \) \( \subseteq \) \( \{x | x = 3n + 1 \text{ where } n \text{ is a natural number}\} \)

12. Indicate which symbol, \( \subseteq \) or \( \not\subseteq \), makes each part of problem 11 true.

---

Mathematical Connections 2-2

Communication

1. Explain the difference between a well-defined set and one that is not. Give examples.

2. Which of the following sets are not well defined? Explain.
   a. the set of wealthy schoolteachers
   b. the set of great books
   c. the set of natural numbers greater than 100
   d. the set of subsets of \( \{1, 2, 3, 4, 5, 6\} \)
   e. the set \( \{x | x \neq x \text{ and } x \in N\} \)

3. Is \( \emptyset \) a proper subset of every nonempty set? Explain your reasoning.

4. Explain why \( \{\emptyset\} \) has \( \emptyset \) as an element and also as a subset.

5. Tell how you would show that \( A \not\subseteq B \).

6. Explain why every set is a subset of itself.

7. Define less than or equal to in a way similar to the definition of less than.

Open-Ended

8. a. Give three examples of sets \( A \) and \( B \) and a universal set \( U \) such that \( A \subseteq B \); find \( \overline{A} \) and \( \overline{B} \).
   b. Based on your observations, conjecture a relationship between \( \overline{B} \) and \( \overline{A} \).
   c. Justify your conjecture in (b) using a Venn diagram.

9. Find an infinite set \( A \) such that
   a. \( \overline{A} \) is finite.
   b. \( \overline{A} \) is infinite.

10. Describe two sets from real-life situations such that it is clear from using one-to-one correspondence, and not from counting, that one set has fewer elements than the other.

Cooperative Learning

11. a. Use a calculator if necessary to estimate the time in years it would take a computer to list all the subsets of \( \{1, 2, 3, \ldots, 64\} \). Assume the fastest computer can list one subset in approximately 1 microsecond (one-millionth of a second).
   b. Estimate the time in years it would take the computer to exhibit all the one-to-one correspondences between the sets \( \{1, 2, 3, \ldots, 64\} \) and \( \{65, 66, 67, \ldots, 128\} \).

12. Using people in your class standing in a line, determine the number of possible arrangements of 1, 2, 3, 4, and 5 people that exist. Use your model to validate the Fundamental Counting Principle.

Questions from the Classroom

13. A student argues that \( \{\emptyset\} \) is the proper notation for the empty set. What is your response?
14. A student claims that a finite set is any set that has a greatest element. Do you agree?
15. A student argues that \( A = \{1, \{1\}\} \) has only one element. How do you respond?
16. A student states that either \( A \subseteq B \) or \( B \subseteq A \). Is the student correct?

Review Problems
17. Investigate the measuring of lengths in the metric system. Develop a plan for using place value with lengths to convert among different metric units.
18. Write 5280 in expanded form.
19. What is the value of MCDX in Hindu-Arabic numerals?
20. Convert each of the following to base ten:
   a. \( 607_{\text{twelve}} \)
   b. \( 1011_{\text{two}} \)
   c. \( 43_{\text{five}} \)
21. If 1 month is approximately 4 weeks and 1 year is approximately 365 days, or 52 weeks, answer the following:
   a. Lewis and Clark spent approximately 2 years, 4 months, and 9 days exploring the territory in the Northwest. What is this time in weeks?
   b. It took Magellan 1126 days to circle the world. How many years is this?
   c. How many seconds old are you?
   d. Approximately how many times does your heart beat in 1 year?

National Assessment of Educational Progress (NAEP) Question
Four people—\( A, X, Y, \) and \( Z \)—go to a movie and sit in adjacent seats. If \( A \) sits in the aisle seat, list all possible arrangements of the other three people. One of the arrangements is shown below.

BRAIN TEASER  Mr. Gonzales’s and Ms. Chan’s seventh-grade classes in Paxson Middle School have 24 and 25 students, respectively. Linda, a student in Mr. Gonzales’s class, claims that the number of school committees that could be formed to contain at least one student from each class is greater than the number of people in the world. Assuming that a committee can have up to 49 students, find the number of committees and determine if Linda is right.

2-3 Other Set Operations and Their Properties

Finding the complement of a set is an operation that acts on only one set at a time. In this section, we consider operations that act on two sets at a time.

Set Intersection
Suppose that during the fall quarter, a college wants to mail a survey to all its students who are enrolled in both art and biology classes. To do this, the school officials must identify those students who are taking both classes. If \( A \) and \( B \) are the set of students taking art courses and the set of students taking biology courses, respectively, during the fall quarter, then the desired set of students includes those common to \( A \) and \( B \), or the intersection of \( A \) and \( B \). The intersection of sets \( A \) and \( B \) is the shaded region in Figure 2-15.

REMARK  Figure 2-15 depicts the possibility of \( A \) and \( B \) containing common elements. The intersection might contain no elements.
Set Union

If \( A \) is the set of students taking art courses during the fall quarter and \( B \) is the set of students taking biology courses during the fall quarter, then the set of students taking art or biology or both during the fall quarter is the union of sets \( A \) and \( B \). The union of sets \( A \) and \( B \) is pictured in Figure 2-17.

Definition of Set Union

The union of two sets \( A \) and \( B \), written \( A \cup B \), is the set of all elements in \( A \) or in \( B \), \( A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \).

The key word in the definition of union is \textit{or} (see Chapter 1). In mathematics, \textit{or} usually means “one or the other or both.” This is known as the inclusive \textit{or}.

Example 2-11

Find \( A \cap B \) in each of the following:

a. \( A = \{1,2,3,4\}, B = \{3,4,5,6\} \)

b. \( A = \{0,2,4,6,\ldots\}, B = \{1,3,5,7,\ldots\} \)

c. \( A = \{2,4,6,8,\ldots\}, B = \{1,2,3,4,\ldots\} \)

Solution

a. \( A \cap B = \{3,4\} \).

b. \( A \cap B = \emptyset \); therefore \( A \) and \( B \) are disjoint.

c. \( A \cap B = A \) because all the elements of \( A \) are also in \( B \).

Set Union

If \( A \) is the set of students taking art courses during the fall quarter and \( B \) is the set of students taking biology courses during the fall quarter, then the set of students taking art or biology or both during the fall quarter is the union of sets \( A \) and \( B \). The union of sets \( A \) and \( B \) is pictured in Figure 2-17.

Definition of Set Union

The union of two sets \( A \) and \( B \), written \( A \cup B \), is the set of all elements in \( A \) or in \( B \), \( A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \).
Example 2.12  Find $A \cup B$ for each of the following:

a. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

b. $A = \{0, 2, 4, 6, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$

c. $A = \{2, 4, 6, 8, \ldots\}$, $B = \{1, 2, 3, 4, \ldots\}$

Solution  

a. $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

b. $A \cup B = \{0, 1, 2, 3, 4, \ldots\}$.

c. Because every element of $A$ is already in $B$, we have $A \cup B = B$.

NOW TRY THIS 2.13  Notice that in Figure 2-16, $n(A \cup B) = 80 + 20 + 180 = 280$, but $n(A) + n(B) = 100 + 200 = 300$; hence in general, $n(A \cup B) \neq n(A) + n(B)$. Use the concept of intersection of sets to write a formula for $n(A \cup B)$.

Set Difference

If $A$ is the set of students taking art classes during the fall quarter and $B$ is the set of students taking biology classes, then the set of all students taking biology but not art is called the complement of $A$ relative to $B$, or the set difference of $B$ and $A$.

Definition of Relative Complement

The complement of $A$ relative to $B$, written $B - A$, is the set of all elements in $B$ that are not in $A$; $B - A = \{x \mid x \in B \text{ and } x \notin A\}$.

Remark  Note that $B - A$ is not read as “$B$ minus $A$.” Minus is an operation on numbers and set difference is an operation on sets.

A Venn diagram representing $B - A$ is shown in Figure 2-18(a). The shaded region represents all the elements that are in $B$ but not in $A$. A Venn diagram for $B \cap \overline{A}$ is given in Figure 2-18(b). The shaded region represents all the elements that are in $B$ and in $\overline{A}$. Notice that $B \cap \overline{A} = B - A$ because $B \cap \overline{A}$ is, by definition of intersection and complement, the set of all elements in $B$ and not in $A$.

Figure 2-18
Example 2-13

If \( A = \{d,e,f\}, B = \{a,b,c,d,e,f\} \), and \( C = \{a,b,c\} \), find each of the following:

a. \( A - B \)
b. \( B - A \)
c. \( B - C \)
d. \( C - B \)
e. To answer parts (a)–(d), does it matter what the universal set is?

Solution

a. \( A - B = \emptyset \)
b. \( B - A = \{a,b,c\} \)
c. \( B - C = \{d,e,f\} \)
d. \( C - B = \emptyset \)
e. Parts (a)–(d) can be answered independently of the universal set. The definition of set difference relates one set to another, independent of the universal set.

Properties of Set Operations

Because the order of elements in a set is not important, \( A \cup B \) is equal to \( B \cup A \). This is the **commutative property of set union**. It does not matter in which order we write the sets when the union of two sets is involved. Similarly, \( A \cap B = B \cap A \). This is the **commutative property of set intersection**.

**NOW TRY THIS 2-14** Use Venn diagrams and other means to find whether grouping is important when the same operation is involved. For example, is it always true that \( A \cap (B \cap C) = (A \cap B) \cap C \)? Similar questions should be investigated involving union and set difference.

In answering Now Try This 2-14, you may have discovered the following properties:

**Theorem 2–2: Associative Property of Set Intersection and Associative Property of Set Union**

The property \( A \cap (B \cap C) = (A \cap B) \cap C \) is the **associative property of set intersection**. Similarly, \( A \cup (B \cup C) = (A \cup B) \cup C \) is the **associative property of set union**.

Example 2-14

Is grouping important when two different set operations are involved? For example, is it true that \( A \cap (B \cup C) = (A \cap B) \cup C \)?

Solution

To investigate this, we let \( A = \{a,b,c,d\}, B = \{c,d,e\}, \) and \( C = \{d,e,f,g\} \). Then

\[
A \cap (B \cup C) = \{a,b,c,d\} \cap (\{c,d,e\} \cup \{d,e,f,g\})
\]

\[
= \{a,b,c,d\} \cap \{c,d,e,f,g\}
\]

\[
= \{c,d\}
\]

\[
(A \cap B) \cup C = (\{a,b,c,d\} \cap \{c,d,e\}) \cup \{d,e,f,g\}
\]

\[
= \{c,d\} \cup \{d,e,f,g\}
\]

\[
= \{c,d,e,f,g\}
\]

In this case, \( A \cap (B \cup C) \neq (A \cap B) \cup C \). So we have found a counterexample, that is, an example illustrating that the general statement is not always true.
To discover an expression that is equal to \( A \cap (B \cup C) \), consider the Venn diagram for \( A \cap (B \cup C) \) shown by the shaded region in Figure 2-19. In the figure, \( A \cap C \) and \( A \cap B \) are subsets of the shaded region. The union of \( A \cap C \) and \( A \cap B \) is the entire shaded region. Thus, \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \). This property is stated formally next.

Theorem 2–3: Distributive Property of Set Intersection over Union

For all sets \( A, B \), and \( C \),
\[
A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
\]

NOW TRY THIS 2-15 If on both sides of the equation in the distributive property of set intersection over union the symbol \( \cap \) is replaced by \( \cup \) and the symbol \( \cup \) is replaced by \( \cap \), is the new property true? Explain why. What should this property be called?

Example 2-15

Use set notation to describe the shaded portions of the Venn diagrams in Figure 2-20.

Solution

The solutions can be described in many different, but equivalent, forms. Some possible answers follow:

a. \((A \cup B) - (A \cap B), (A \cup B) \cap (A \cap B), \) or \((A - B) \cup (B - A)\)

b. \((A \cap B) \cup (B \cap C) \) or \(B \cap (A \cup C)\)

c. \((A - B) - C, A - (B \cup C), \) or \((A - (A \cap B)) - (A \cap C)\)

d. \(((A \cup C) - B) \cup (A \cap B \cap C) \) or \((A - (B \cup C)) \cup (C - (A \cup B)) \cup (A \cap C)\)

Using Venn Diagrams as a Problem-Solving Tool

Venn diagrams can be used as a problem-solving tool for modeling information, as shown in the following examples.
Example 2-16

Suppose $M$ is the set of all students taking mathematics and $E$ is the set of all students taking English. Identify the students described by each region in Figure 2-21.

Solution

Region (a) contains all students taking mathematics but not English.
Region (b) contains all students taking both mathematics and English.
Region (c) contains all students taking English but not mathematics.
Region (d) contains all students taking neither mathematics nor English.

Example 2-17

In a survey of 110 college freshmen that investigated their high school backgrounds, the following information was gathered:

- 25 took physics.
- 45 took biology.
- 48 took mathematics.
- 10 took physics and mathematics.
- 8 took biology and mathematics.
- 6 took physics and biology.
- 5 took all three subjects.

a. How many students took biology but neither physics nor mathematics?
b. How many took physics, biology, or mathematics?
c. How many did not take any of the three subjects?

Solution

To solve this problem, we build a model using sets. Because there are three distinct subjects, we should use three circles. In Figure 2-22, $P$ is the set of students taking physics, $B$ is the set taking biology, and $M$ is the set taking mathematics. The shaded region represents the 5 students who took all three subjects. The lined region represents the students who took physics and mathematics, but who did not take biology.

In part (a) we are asked for the number of students in the subset of $B$ that has no element in common with either $P$ or $M$. That is, $B - (P \cup M)$. In part (b) we are asked for the number of elements in $P \cup B \cup M$. Finally, in part (c) we are asked for the number of students in $P \cup B \cup M$, or $U - (P \cup B \cup M)$. Our strategy is to find the number of students in each of the eight nonoverlapping regions.

One mindset to beware of in this problem is thinking, for example, that the 25 students who took physics, took only physics. That is not necessarily the case. If those students had been taking only physics, then we should have been told so.

a. Because a total of 10 students took physics and mathematics and 5 of those also took biology, $10 - 5$, or 5, students took physics and mathematics but not biology. Similarly,
For a class project, students collect data about the number of boys or girls in the families of their classmates. Use the table below to answer Exercises 27–30.

<table>
<thead>
<tr>
<th>Name</th>
<th>Number of Boys in the Family</th>
<th>Number of Girls in the Family</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anya</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Brian</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Charlie</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Diane</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Elisha</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Felix</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Gloria</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Han</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Ivan</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Jorge</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

27. Anya wants to make a Venn diagram with the groups “Has Boys in the Family” and “Has Girls in the Family.” She begins by placing herself on the diagram. Copy and complete her diagram below.

![Venn Diagram](image)

28. Make a bar graph that shows that the usual number of children in the family is two and that Brian’s family is unusual.

29. Charlie wants to make a circle graph. What could the parts of the circle be labeled?

30. Make a graph to see if there is any relationship between the number of boys in a family and the number of girls in a family.
because 8 students took biology and mathematics and 5 took all three subjects, \( 8 - 5 \), or 3, took biology and mathematics but not physics. Also \( 6 - 5 \), or 1, student took physics and biology but not mathematics. To find the number of students who took biology but neither physics nor mathematics, we subtract from 45 (the total number that took biology) the number of those that are in the distinct regions that include biology and other subjects, that is, \( 1 + 5 + 3 \), or 9. Because \( 45 - 9 = 36 \), we know that 36 students took biology but neither physics nor mathematics.

b. To find the number of students in all the distinct regions in \( P, M, \) or \( B \), we proceed as follows. The number of students who took physics but neither mathematics nor biology is \( 25 - (1 + 5 + 5) \), or 14. The number of students who took mathematics but neither physics nor biology is \( 48 - (5 + 5 + 3) \), or 35. Hence the number of students who took mathematics, physics, or biology is \( 35 + 14 + 36 + 3 + 5 + 5 + 1 \), or 99.

c. Because the total number of students is 110, the number that did not take any of the three subjects is \( 110 - 99 \), or 11.

**Cartesian Products**

Another way to produce a set from two given sets is by forming the **Cartesian product**. This formation pairs the elements of one set with the elements of another set in a specific way to create elements in a new set. Suppose a person has three pairs of pants, \( P = \{ \text{blue, white, green} \} \), and two shirts, \( S = \{ \text{blue, red} \} \). According to the Fundamental Counting Principle, there are \( 3 \cdot 2 \), or 6, possible different pant-and-shirt pairs, as shown in Figure 2-23.

![Figure 2-23](image-url)

The pants-shirt combinations are elements of the set of all possible pairs in which the first member of the pair is an element of set \( P \) and the second member is an element of set \( S \). The set of all possible pairs is given in Figure 2-23. Because the first component in each pair represents pants and the second component in each pair represents shirts, the order in which the components are written is important. Thus (green, blue) represents green pants and a blue shirt, whereas (blue, green) would represent blue pants and a green shirt. Therefore, the two pairs represent different outfits. Because the order in each pair is important, the pairs are **ordered pairs**. The positions that the ordered pairs occupy within the set of outfits is immaterial. Only the order of the **components** within each pair is significant.

The pants-and-shirt pairs suggest the following definition of **equality for ordered pairs**: \( (x,y) = (m,n) \) if, and only if, the first components are equal and the second components are equal. A set consisting of ordered pairs is an example of a Cartesian product. A formal definition follows.
Definition of Cartesian Product

For any sets $A$ and $B$, the **Cartesian product** of $A$ and $B$, written $A \times B$, is the set of all ordered pairs such that the first component of each pair is an element of $A$ and the second component of each pair is an element of $B$.

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

**Remark** $A \times B$ is commonly read as “$A$ cross $B$” and should never be read “$A$ times $B$.”

**Example 2-18**

If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, find each of the following:

a. $A \times B$

b. $B \times A$

c. $A \times A$

**Solution**

a. $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$

b. $B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

c. $A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

It is possible to form a Cartesian product involving the null set. Suppose $A = \{1, 2\}$. Because there are no elements in $\emptyset$, no ordered pairs $(x, y)$ with $x \in A$ and $y \in \emptyset$ are possible, so $A \times \emptyset = \emptyset$. This is true for all sets $A$. Similarly, $\emptyset \times A = \emptyset$ for all sets $A$. There is an analogy between the last equation and the multiplication fact that $0 \cdot a = 0$, where $a$ is a natural number. In Chapter 3 we use the concept of Cartesian product to define multiplication of natural numbers.

**Assessment 2-3A**

1. If $N = \{1, 2, 3, 4, \ldots\}$, $A = \{x \mid x = 2n - 1 \text{ where } n \in N\}$, $B = \{x \mid x = 2n \text{ where } n \in N\}$, and $C = \{x \mid x = 2n + 1 \text{ where } n = 0 \text{ or } n \in N\}$, find the simplest possible expression for each of the following:
   a. $A \cup C$
   b. $A \cup B$
   c. $A \cap B$

2. Decide whether the following pairs of sets are always equal:
   a. $A \cap B$ and $B \cap A$
   b. $A \cup B$ and $B \cup A$
   c. $A \cup (B \cup C)$ and $(A \cup B) \cup C$
   d. $A \cap A$ and $A \cup \emptyset$

3. Tell whether each of the following is true for all sets $A$ and $B$. If false, give a counterexample.
   a. $A \cup \emptyset = A$
   b. $A - B = B - A$
   c. $A \cap B = B \cap A$
   d. $(A \cup B) - A = B$
   e. $(A - B) \cup A = (A \cap B) \cup (B - A)$

4. If $B \subseteq A$, find a simpler expression for each of the following:
   a. $A \cap B$
   b. $A \cup B$

5. For each of the following, shade the portion of the Venn diagram that illustrates the set:
   a. $A \cup B$
   b. $A \cap B$
   c. $(A \cap B) \cup (A \cap C)$
   d. $(A \cup B) \cap C$
   e. $(A \cap B) \cup C$

6. If $S$ is a subset of universe $U$, find each of the following:
   a. $S \cup \emptyset$
   b. $U$
   c. $S \cap \emptyset$
   d. $\emptyset \cap S$

7. For each of the following conditions, find $A - B$:
   a. $A \cap B = \emptyset$
   b. $B = U$

8. If for sets $A$ and $B$ we know that $A - B = \emptyset$, is it necessarily true that $A \subseteq B$? Justify your answer.
9. Use set notation to identify each of the following shaded regions:

![Venn Diagram A and B](image)

- **a.** $A \cap B$
- **b.** $A \cup B$
- **c.** $A \cap B$ (not shaded)

10. In the following, shade the portion of the Venn diagram that represents the given set:

![Venn Diagram A and B](image)

- **a.** $A \cap \bar{B}$

11. Use Venn diagrams to determine if each of the following is true:

- **a.** $A \cup (B \cap C) = (A \cup B) \cap C$
- **b.** $A - (B - C) = (A - B) - C$

12. For each of the following pairs of sets, explain which is a subset of the other. If neither is a subset of the other, explain why:

- **a.** $A \cap B$ and $A \cap B \cap C$
- **b.** $A \cup B$ and $A \cup B \cup C$

13. **a.** If $A$ has three elements and $B$ has two elements, what is the greatest number of elements possible in $i) A \cup B$?

- **ii) $A \cap B$**
- **iii) $B - A$**
- **iv) $A - B$**

- **b.** If $A$ has $n$ elements and $B$ has $m$ elements, what is the greatest number of elements possible in $i) A \cup B$?

- **ii) $A \cap B$**
- **iii) $B - A$**
- **iv) $A - B$**

14. If $n(A) = 4, n(B) = 5$, and $n(C) = 6$, what is the greatest and least number of elements possible in:

- **a.** $A \cup B \cup C$?
- **b.** $A \cap B \cap C$?

15. Given that the universe is the set of all humans, $B = \{ x \mid x$ is a college basketball player $\}$, and $S = \{ x \mid x$ is a college student more than 200 cm tall $\}$, describe each of the following in words:

- **a.** $B \cap S$
- **b.** $S$
- **c.** $B \cup S$
- **d.** $B \cup S$
- **e.** $B \cap S$
- **f.** $B \cap S$

16. Of the eighth graders at the Paxson School, 7 played basketball, 9 played volleyball, 10 played soccer, 1 played basketball and volleyball only, 1 played basketball and soccer only, 2 played volleyball and soccer only, and 2 played volleyball, basketball, and soccer. How many played one or more of the three sports?

17. In a fraternity with 30 members, 18 take mathematics, 5 take both mathematics and biology, and 8 take neither mathematics nor biology. How many take biology but not mathematics?

18. In Paul’s bicycle shop, 40 bicycles are inspected. If 20 needed new tires and 30 needed gear repairs, answer the following:

- **a.** What is the greatest number of bikes that could have needed both?
- **b.** What is the least number of bikes that could have needed both?
- **c.** What is the greatest number of bikes that could have needed neither?

19. The Red Cross looks for three types of antigens in blood tests: $A$, $B$, and $Rh$. When the antigen $A$ or $B$ is present, it is listed, but if both these antigens are absent, the blood is type $O$. If the Rh antigen is present, the blood is positive; otherwise, it is negative. If a laboratory technician reports the following results after testing the blood samples of 100 people, how many were classified as $O$ negative? Explain your reasoning.

<table>
<thead>
<tr>
<th>Number of Samples</th>
<th>Antigen in Blood</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>A</td>
</tr>
<tr>
<td>18</td>
<td>B</td>
</tr>
<tr>
<td>82</td>
<td>Rh</td>
</tr>
<tr>
<td>5</td>
<td>A and B</td>
</tr>
<tr>
<td>31</td>
<td>A and Rh</td>
</tr>
<tr>
<td>11</td>
<td>B and Rh</td>
</tr>
<tr>
<td>4</td>
<td>A, B, and Rh</td>
</tr>
</tbody>
</table>

20. Classify the following as true or false. If false, give a counterexample. Assume that $A$ and $B$ are finite sets.

- **a.** If $n(A) = n(B)$, then $A = B$.
- **b.** If $A = B = \emptyset$, then $A = B$.
- **c.** If $A \subset B$, where $A$ and $B$ are finite, then $n(A) < n(B)$.

21. Three announcers each tried to predict the winners of Sunday’s professional football games. The only team not picked that is playing Sunday was the Giants. The choices for each person were as follows:

- **Phyllis:** Cowboys, Steelers, Vikings, Bills
- **Paula:** Steelers, Packers, Cowboys, Redskins
- **Rashid:** Redskins, Vikings, Jets, Cowboys
Let \( A = \{x,y\} \) and \( B = \{a,b,c\} \). Find each of the following:

- a. \( A \times B \)
- b. \( B \times A \)

23. For each of the following, the Cartesian product \( C \times D \) is given by the sets listed. Find \( C \) and \( D \).

- a. \( \{(a,b),(a,c),(a,d),(a,e)\} \)
- b. \( \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\} \)
- c. \( \{(0,1),(0,0),(1,1),(1,0)\} \)

Assessment 2-3B

1. If \( W = \{0,1,2,3,\ldots\} \), \( A = \{x \mid x = 2n + 1 \text{ where } n \in W\} \), \( B = \{x \mid x = 2n \text{ where } n \in W\} \), and \( N = \{1,2,3,\ldots\} \), find the simplest possible expression for each of the following:

- a. \( W - A \)
- b. \( A \cap B \)
- c. \( W \cap N \)

2. Decide whether the following pairs of sets are always equal.

- a. \( X \cap Y \) and \( Y \cap X \)
- b. \( X \cup Y \) and \( Y \cup X \)
- c. \( A \cap (B \cap C) \) and \( (A \cap B) \cap C \)
- d. \( B \cup \emptyset \) and \( B \cap B \)

3. Tell whether each of the following is true for all sets \( A, B, \) or \( C \). If false, give a counterexample.

- a. \( A - B = A \cap B \)
- b. \( A \cup B = A \cup B \)
- c. \( A \cap (B \cup C) = (A \cap B) \cup C \)
- d. \( (A - B) \cap A = A \cap B \)
- e. \( A - (B \cap C) = (A - B) \cap (A - C) \)

4. If \( X \subseteq Y \), find a simpler expression for each of the following:

- a. \( X - Y \)
- b. \( X \cap Y \)

5. For each of the following, shade the portion of the Venn diagram that illustrates the set:

- a. \( A \cap \overline{C} \)
- b. \( A \cup B \)
- c. \( (A \cap B) \cup (B \cap C) \)
- d. \( A \cup (B \cap C) \)
- e. \( A \cup (B \cap C) \)

6. If \( A \) is a subset of universe \( U \), find each of the following:

- a. \( A \cup U \)
- b. \( U - A \)
- c. \( A - \emptyset \)
- d. \( \emptyset \cap A \)

7. For each of the following conditions, find \( B - A \).

- a. \( A = B \)
- b. \( B \subseteq A \)

8. Give two examples of sets \( A \) and \( B \) for which \( B - A = \emptyset \). Show that in each example \( B \subseteq A \).

9. Use set notation to identify each of the following shaded regions:

10. In the following, shade the portion of the Venn diagram that represents the given set:

11. Use Venn diagrams to determine if each of the following is true:

- a. \( A - (B \cap C) = (A - B) \cap (A - C) \)
- b. \( A - (B \cup C) = (A - B) \cup (A - C) \)

12. For each of the following pairs of sets, explain which is a subset of the other. If neither is a subset of the other, explain why.

- a. \( A - B \) and \( A - (B - C) \)
- b. \( A \cup B \) and \( (A \cup B) - \emptyset \)

13. a. If \( n(A \cup B) = 22 \) and \( n(A \cap B) = 8 \), and \( n(B) = 12 \), find \( n(A) \).

- b. If \( n(A) = 8 \), \( n(B) = 14 \), and \( n(A \cap B) = 5 \), find \( n(A \cup B) \).

14. The equation \( \overline{A} \cap B = A \cap \overline{B} \) and a similar equation for \( \overline{A} \cup B \) are referred to as DeMorgan's Laws in honor of the famous British mathematician who first discovered them.

- a. Use Venn diagrams to show that \( \overline{A} \cup B = \overline{A} \cap B \).
b. Discover an equation similar to the one in part (a) involving \( A \cap B, A, \) and \( B \). Use Venn diagrams to show that the equation holds.

c. Verify the equations in (a) and (b) for specific sets.

15. Suppose \( P \) is the set of all eighth-grade students at the Paxson School, with \( B \) the set of all students in the band and \( C \) the set of all students in the choir. Identify in words the students described by each region of the following figure:

16. Fill in the Venn diagram with the appropriate numbers based on the following information:

\[
\begin{align*}
n(A) &= 26 & n(B \cap C) &= 12 \\
n(B) &= 32 & n(A \cap C) &= 8 \\
n(C) &= 23 & n(A \cap B \cap C) &= 3 \\
n(A \cap B) &= 10 & n(U) &= 65
\end{align*}
\]

17. Write the letters in the appropriate sections of the following Venn diagram using the following information:

Set \( A \) contains the letters in the word Iowa.
Set \( B \) contains the letters in the word Hawaii.
Set \( C \) contains the letters in the word Ohio.

The universal set \( U \) contains the letters in the word Washington.

18. When three sets \( A, B, \) and \( C \) intersect as in the diagram of problem 17, eight nonoverlapping regions are created. Describe each of the regions using set notation.

19. A pollster interviewed 500 university seniors who owned credit cards. She reported that 240 owned Goldcard, 290 had Supercard, and 270 had Thriftcard. Of those seniors, the report said that 80 owned only a Goldcard and a Supercard, 70 owned only a Goldcard and a Thriftcard, 60 owned only a Supercard and a Thriftcard, and 50 owned all three cards. When the report was submitted for publication in the local campus newspaper, the editor refused to publish it, claiming the poll was not accurate. Was the editor right? Why or why not?

20. A baseball manager examined his roster and noticed the following:

- Every outfielder was a switch hitter.
- A third of the infielders were switch hitters.
- Half of all the switch hitters were outfielders.
- There are 12 infielders and 8 outfielders and no person played both positions.

How many switch hitters are neither infielders nor outfielders?

21. On the first day of tryouts for Little League, 128 boys of ages 10 (\( T \)), 11 (\( E \)), and 12 (\( W \)) showed up. They were asked what positions besides pitcher they wanted to play: infield (\( I \)), outfield (\( O \)), or catcher (\( C \)). The results are shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>( I )</th>
<th>( O )</th>
<th>( C )</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>28</td>
<td>14</td>
<td>12</td>
<td>54</td>
</tr>
<tr>
<td>( E )</td>
<td>18</td>
<td>20</td>
<td>8</td>
<td>46</td>
</tr>
<tr>
<td>( W )</td>
<td>10</td>
<td>12</td>
<td>6</td>
<td>28</td>
</tr>
<tr>
<td>Totals</td>
<td>56</td>
<td>46</td>
<td>26</td>
<td>128</td>
</tr>
</tbody>
</table>

Tell what each of the following means in words along with the number of boys indicated in each part:

a. \( I \cap W \)

b. \( C \cap (T \cup E) \)

c. \((I \cup O) \cap T \)

d. \((T \cup E) \cap O \)

22. Tell whether each of the following is true or false and tell why:

a. \((2,5) = (5,2)\)

b. \((2,5) = \{2,5\}\)

23. Answer each of the following:

a. If \( A \) has five elements and \( B \) has four elements, how many elements are in \( A \times B \)?

b. If \( A \) has \( m \) elements and \( B \) has \( n \) elements, how many elements are in \( A \times B \)?

c. If \( A \) has \( m \) elements, \( B \) has \( n \) elements, and \( C \) has \( p \) elements, how many elements are in \( (A \times B) \times C \)?
Section 2-3  Other Set Operations and Their Properties  105

Communication

1. Answer each of the following and justify your answer:
   a. If \( a \in A \cap B \), is it true that \( a \in A \cup B \)?
   b. If \( a \in A \cup B \), is it true that \( a \in A \cap B \)?
2. Explain how \( \bar{A} \) is related to \( U - A \).
3. Is the operation of forming Cartesian products commutative? Explain why or why not.
4. If \( A \) and \( B \) are sets, is it always true that \( n(A - B) = n(A) - n(B) \)? Explain.

Open-Ended

5. Make up and solve a story problem concerning specific sets \( A \), \( B \), and \( C \) for which \( n(A \cup B \cup C) \) is known and it is required to find \( n(A) \), \( n(B) \), and \( n(C) \).
6. Describe a real-life situation that can be represented by each of the following:
   a. \( A \cap B \)
   b. \( A \cap B \cap C \)
   c. \( A - (B \cup C) \)

Cooperative Learning

7. Use set operations of union, intersection, complement, and set difference to describe the shaded region in the following figure in as many ways as possible. Compare your expressions with those of other groups to see which has the most. What is the total number of different expressions found by all the groups? Which expressions appeared in all the groups?

Questions from the Classroom

8. A student asks, “If \( A = \{a, b, c\} \) and \( B = \{b, c, d\} \), why isn’t it true that \( A \cup B = \{a, b, c, d\}? \)” What is your response?
9. A student says that she can show that if \( A \cap B = A \cap C \), then it is not necessarily true that \( B = C \); but she thinks that whenever \( A \cap B = A \cap C \) and \( A \cup B = A \cup C \), then \( B = C \). What is your response?
10. A student claims that the complement bar can be “broken” over the operation of intersections; that is, \( \bar{A \cap B} = \bar{A} \cap \bar{B} \). What is your response?
11. A student is asked to find all one-to-one correspondences between two given sets. He finds the Cartesian product of the sets and claims that his answer is correct because it includes all possible pairings between the elements of the sets. How do you respond?
12. A student argues that adding two sets \( A \) and \( B \), or \( A + B \), and taking the union of two sets, \( A \cup B \), is the same thing. How do you respond?

Review Problems

13. In base two, does the number “two” exist? Explain your reasoning.
14. How would you write 81 in base three? Any power of 3 in base ten?
15. a. Write \( \{4, 5, 6, 7, 8, 9\} \) using set-builder notation.
    b. Write \( \{x \mid x = 5n, \text{where } n = 3, 6, \text{or } 9\} \) using the listing method.
16. Find the number of elements in the following sets:
   a. \( \{x \mid x \text{ is a letter in \textit{common sense}}\} \)
   b. The set of letters appearing in the word \textit{committee}
17. If \( A = \{1, 2, 3, 4\} \) and \( B = \{1, 2, 3, 4, 5\} \), answer the following questions:
   a. How many subsets of \( A \) do not contain the element 1?
   b. How many subsets of \( A \) contain the element 1?
   c. How many subsets of \( A \) contain either the element 1 or 2?
   d. How many subsets of \( A \) contain neither the element 1 nor 2?
   e. How many subsets of \( B \) contain the element 5 and how many do not?
   f. If all the subsets of \( A \) are known, how can all the subsets of \( B \) be listed systematically? How many subsets of \( B \) are there?
18. a. Which of the following sets are equal?
    b. Which sets are proper subsets of the other sets?
       \[ A = \{2, 4, 6, 8, 10, \ldots\} \]
       \[ B = \{x \mid x = 2n + 2 \text{ where } n = 0, 1, 2, 3, 4, \ldots\} \]
       \[ C = \{x \mid x = 4n \text{ where } n \in \mathbb{N}\} \]
19. Give examples from real life for each of the following:
   a. A one-to-one correspondence between two sets
   b. A correspondence between two sets that is not one-to-one
20. If there are six teams in the Alpha league and five teams in the Beta league and if each team from one league plays each team from the other league exactly once, how many games are played?

21. José has four pairs of slacks, five shirts, and three sweaters. From how many combinations can he choose if he chooses a pair of slacks, a shirt, and a sweater each day?

National Assessment of Educational Progress (NAEP) Question

Melissa chose one of the figures above.
- The figure she chose was shaded.
- The figure she chose was not a triangle.
Which figure did she choose?


LABORATORY ACTIVITY  A set of attribute blocks consists of 32 blocks. Each block is identified by its own shape, size, and color. The four shapes in a set are square, triangle, rhombus, and circle; the four colors are red, yellow, blue, and green; the two sizes are large and small. In addition to the blocks, each set contains a group of 20 cards. Ten of the cards specify one of the attributes of the blocks (for example, red, large, square). The other 10 cards are negation cards and specify the lack of an attribute (for example, not green, not circle). Many set-type problems can be studied with these blocks. For example, let A be the set of all green blocks and B be the set of all large blocks. Using the set of all blocks as the universal set, describe elements in each set listed here to determine which are equal:

1. $A \cup B; B \cup A$
2. $A \cap B; A \cap \overline{B}$
3. $A \cap \overline{B}; A \cup \overline{B}$
4. $A \setminus B; A \setminus \overline{B}$

Hint for Solving the Preliminary Problem

The Venn diagram of this chapter is a good tool for sorting the data. Try distinguishing among disjoint sets of people. For example, consider one circle with adults, one with Mississippi citizens, and one with females. Sorting the information with these circles should provide guidance in finding a solution. Remember to determine what types of people are in the intersections and what the complements of the sets are.

Chapter Outline

I. Numeration systems
   A. A numeration system consists of a set of symbols with operations and properties to represent numbers systematically.
   B. Properties of numeration systems give basic structure to the systems.
      1. Additive property
      2. Place value property
      3. Subtractive property

   4. Multiplicative property

   5. Place value assigns a value to a digit depending on its placement in a numeral. The value of a digit is the product of its place value and its face value.

   C. The Hindu-Arabic numeration system is a base-ten system that uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
II. Exponents
A. For any whole number $a$ and any natural number $n$,

$$a^n = a \cdot a \cdot a \cdot \ldots \cdot a$$

$n$ factors

where $a$ is the base and $n$ is the exponent.
B. $a^0 = 1$ where $a \in N$

III. Set definitions and notation
A. A set can be described as any collection of objects.
B. Sets should be well defined so that an object either does or does not belong to the set.
C. An element is any member of a set.
D. Sets can be specified by either listing all the elements or using set-builder notation.
E. The empty set, written $\emptyset$, contains no elements.
F. The universal set contains all the elements being discussed.

IV. Relationships and operations on sets
A. Two sets are equal if, and only if, they have exactly the same elements.
B. Two sets $A$ and $B$ are in one-to-one correspondence if, and only if, each element of $A$ can be paired with exactly one element of $B$ and each element of $B$ can be paired with exactly one element of $A$.
C. Two sets $A$ and $B$ are equivalent if, and only if, their elements can be placed into one-to-one correspondence (written $A \sim B$).
D. Set $A$ is a subset of set $B$ if, and only if, every element of $A$ is an element of $B$ (written $A \subseteq B$).
E. Set $A$ is a proper subset of set $B$ if, and only if, every element of $A$ is an element of $B$ and there is at least one element of $B$ that is not in $A$ (written $A \subset B$).

F. A set containing $n$ elements has $2^n$ subsets.
G. The union of two sets $A$ and $B$ is the set of all elements in $A$, in $B$, or in both $A$ and $B$ (written $A \cup B$).
H. The intersection of two sets $A$ and $B$ is the set of all elements belonging to both $A$ and $B$ (written $A \cap B$).
I. The cardinal number of a finite set $S$, $n(S)$, indicates the number of elements in the set.
J. A set is finite if the number of elements in the set is zero or a natural number. Otherwise, the set is infinite.
K. Two sets $A$ and $B$ are disjoint if they have no elements in common.
L. The complement of a set $A$ is the set consisting of the elements of the universal set not in $A$ (written $\complement A$).
M. The complement of set $A$ relative to set $B$ (set difference) is the set of all elements in $B$ that are not in $A$ (written $B - A$).
N. The Cartesian product of sets $A$ and $B$, written $A \times B$, is the set of all ordered pairs such that the first element in each pair is from $A$ and the second element of each pair is from $B$.

O. Properties of set operations
1. Commutative properties of set union and intersection
2. Associative properties of set union and intersection
3. Distributive property of set intersection over union and of set union over intersection

P. Fundamental Counting Principle: If event $M$ can occur in $m$ ways and, after it has occurred, event $N$ can occur in $n$ ways, then event $M$ followed by event $N$ can occur in $mn$ ways.
8. a. The first digit from the left (the lead digit) of a base-ten numeral is 4 followed by 10 zeros. What is the place value of 4?
   b. A number in base five has 10 digits. What is the place value of the second digit from the left?
   c. A number in base two has lead digit 1 followed by 30 zeros and units digit 1. What is the place value of the lead digit?
9. Write the following base-ten numerals in the indicated base without performing any multiplications:
   a. $10^{10} + 23$ in base ten
   b. $2^{10} + 1$ in base two
   c. $5^{10} + 1$ in base five
   d. $10^{10} - 1$ in base ten
   e. $2^{10} - 1$ in base two
   f. $12^5 - 1$ in base twelve
10. Write an example of a base other than ten used in a real-life situation. How is it used?
11. Describe the important characteristics of each of the following systems:
   a. Egyptian
   b. Babylonian
   c. Roman
   d. Hindu-Arabic
12. Write 128 in each of the following bases:
   a. five
   b. two
   c. twelve
13. Write each of the following in the indicated bases without multiplying out the various powers:
   a. $4 \cdot 5^6 + 11 \cdot 5^3 + 9$ in base five
   b. $2^{10} + 2^3$ in base two
   c. $11 \cdot 12^3 + 10 \cdot 12^3 + 20$ in base twelve
   d. $9 \cdot 8^3 + 8$ in base eight
14. List all the subsets of \{m, a, t, b\}.
15. Let $U = \{a, n, i, v, e, r, s, a, l\}$, $A = \{r, a, v, e\}$, $C = \{i, n, v, e\}$, $B = \{a, r, e\}$, $D = \{s, a, i, e\}$. Find each of the following:
   a. $A \cup B$
   b. $C \cap D$
   c. $\overline{D}$
   d. $A \cap \overline{D}$
   e. $B \cup C$
   f. $(B \cup C) \cap D$
   g. $(A \cup B) \cap (C \cap D)$
   h. $(C \cap D) \cap A$
   i. $n(C)$
   j. $n(C \times D)$
16. Indicate the following sets by shading the figure:
   a. $A \cap (B \cup C)$
   b. $(A \cup B) \cap C$
17. Suppose you are playing a word game with seven distinct letters. How many seven-letter words can there be?
18. a. Show one possible one-to-one correspondence between sets $D$ and $E$ if $D = \{t, b, e\}$ and $E = \{e, r, a\}$.
   b. How many one-to-one correspondences between sets $D$ and $E$ are possible?
19. Use a Venn diagram to determine whether $A \cap (B \cup C) = (A \cap B) \cup C$ for all sets $A$, $B$, and $C$.
20. According to a student survey, 16 students liked history, 19 liked English, 18 liked mathematics, 8 liked mathematics and English, 5 liked history and English, 7 liked history and mathematics, 3 liked all three subjects, and every student liked at least one of the subjects. Draw a Venn diagram describing this information and answer the following questions:
   a. How many students were in the survey?
   b. How many students liked only mathematics?
   c. How many students liked English and mathematics but not history?
21. Describe, using symbols, the shaded portion in each of the following figures:
   a. 
   b. 
22. Classify each of the following as true or false. If false, tell why.
   a. For all sets $A$ and $B$, either $A \subseteq B$ or $B \subseteq A$.
   b. The empty set is a proper subset of every set.
   c. For all sets $A$ and $B$, if $A \sim B$, then $A = B$.
   d. The set \{5, 10, 15, 20, \ldots\} is a finite set.
   e. No set is equivalent to a proper subset of itself.
   f. If $A$ is an infinite set and $B \subseteq A$, then $B$ also is an infinite set.
   g. For all finite sets $A$ and $B$, if $A \cap B \neq \emptyset$, then $n(A \cup B) \neq n(A) + n(B)$.
   h. If $A$ and $B$ are sets such that $A \cap B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.
23. Use Venn diagrams to decide whether each of the following is always true for finite sets $A$ and $B$.
   a. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
   b. $n(A \cup B) = n(A - B) + n(B) - n(B - A) - n(A)$
24. Suppose $P$ and $Q$ are equivalent sets and $n(P) = 17$.
   a. What is the minimum number of elements in $P \cup Q$?
   b. What is the maximum number of elements in $P \cup Q$?
c. What is the minimum number of elements in \(P \cap Q\)?

d. What is the maximum number of elements in \(P \cap Q\)?

25. Case Eastern Junior College awarded 26 varsity letters in crew, 15 in swimming, and 16 in soccer. If awards went to 46 students and only 2 lettered in all sports, how many students lettered in just two of the three sports?

26. Consider the set of northwestern states or provinces (Montana, Washington, Idaho, Oregon, Alaska, British Columbia, Alberta). If a person chooses one element, show that in three yes or no questions, we can determine the element.

27. Using the definitions of less than or greater than, prove that each of the following inequalities is true:

   a. \(3 < 13\)
   b. \(12 > 9\)

28. Heidi has a brown pair and a gray pair of slacks; a brown blouse, a yellow blouse, and a white blouse; and a blue sweater and a white sweater. How many different outfits does she have if each outfit she wears consists of slacks, a blouse, and a sweater?