Annie, a fifth-grade teacher, asks every student in her class to think of a number, multiply the number by 6, add 4, then divide the result by 2, add 5, multiply the new result by 2, and then subtract 18. She then asks each student for the final result, and as they answer, tells each one the initial number the student chose. How was Annie able to find each student’s number so quickly?
Because algebraic thinking is so important in mathematics at all levels—from the early grades on—we include a separate chapter on the subject. In this chapter, we will focus not only on patterns (introduced in Chapter 1) but on other features of algebraic thinking as well, including solving equations, word problems, functions, and graphing.

In years past, schoolchildren were not introduced to algebra until at least late middle school. Today, however, we realize the importance of integrating algebraic thinking and problem solving at all levels, beginning with kindergarten. In fact, as the research note points out, algebraic thinking must be taught to all students.

*Principles and Standards* recommends that students in pre-K–2 be able to:

- use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations;
- model situations that involve the addition and subtraction of whole numbers, using objects, pictures, and symbols. (p. 90)

And that in grades 3–5 students be able to:

- represent and analyze patterns and functions, using words, tables, and graphs;
- represent the idea of a variable as an unknown quantity using a letter or a symbol;
- express mathematical relationships using equations;
- investigate how a change in one variable relates to a change in a second variable;
- identify and describe situations with constant or varying rates of change and compare them. (p. 158)

And that students in grades 6–8 be able to:

- identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations;
- develop an initial conceptual understanding of different uses of variables;
- explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope;
- use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships;
- recognize and generate equivalent forms for simple algebraic expressions and solve linear equations. (p. 222)

*Focal Points* states that students in grade 6 should be taught to:

- write mathematical expressions and equations that correspond to given situations
- evaluate expressions
- use expressions and formulas to solve problems
- understand that variables represent numbers whose exact values are not yet specified
- use variables appropriately

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Research Note

We have found that elementary children can learn to engage in algebraic reasoning. Furthermore, learning the big ideas and practices of mathematics is not just for a few mathematically gifted students. In fact, a strong case can be made that it is most critical for students at risk of failing in mathematics to engage with these ideas and practices (Carpenter et al. 2003).
• understand that expressions in different forms can be equivalent
• rewrite an expression to represent a quantity in a different way
• know that the solutions of an equation are the values of the variables that make the equation true
• solve simple one-step equations by using number sense, properties of operations, and the idea of maintaining equality on both sides of an equation
• construct and analyze tables
• use equations to describe simple relationships (such as $3x = y$) shown in a table (p. 18)

In this chapter, we use the basic knowledge of operations to build tenets of algebraic thinking. Subsequent chapters take a closer look at the mathematics assumed here as well as at new mathematics and will delve more deeply into algebraic thinking.

Algebra is a branch of mathematics in which symbols—usually letters—represent numbers or members of a given set. Elementary algebra is used to generalize arithmetic. For example, the fact that $7 + (3 + 5) = (7 + 3) + 5$, or that $9 + (3 + 8) = (9 + 3) + 8$, are special cases of $a + (b + c) = (a + b) + c$, where $a$, $b$, and $c$ are numbers from a given set, for example, whole numbers, integers, rational numbers, or real numbers. Similarly, $2 + 3 = 3 + 2$ and $2 \cdot 3 = 3 \cdot 2$ are special cases of $a + b = b + a$ and $a \cdot b = b \cdot a$ for all whole numbers $a$ and $b$.

The word algebra—the Latinized version of the Arabic word al-jabr—comes from the book *Hidab al-jabr wa’t muqabalah*, written by Mohammed ibn Musa al-Khowarizmi (ca. 825 CE). Al-Khowarizmi (from whose name we get the word algorithm) was part of the House of Wisdom (Bayt al-Hikma), an institution for education and research founded by the caliph al-Ma’mun. In his book he synthesized previous Hindu work on the notions of algebra and used the words *jabr* and *muqabalah* to designate two basic operations in solving equations: *jabr* meant to transpose subtracted terms to the other side of the equation; *muqabalah* meant to cancel like terms on opposite sides of the equation. The title of his book translates as *The Science of Restoring What Is Missing and Equating Like With Like*.

Another major contributor to the development of algebra was Diophantus (ca. 200–284 CE). The *Arithmetica* is the major work of Diophantus and the most prominent work on algebra in Greek mathematics. Of the original thirteen books of which *Arithmetica* consisted, only six have survived.

About 900 years later, Leonardo di Pisa (ca. 1170–1250) introduced algebra to Europe. He was also known as Fibonacci, which means the son of Bonacci. Fibonacci was the greatest mathematician of his age; he made mathematics more accessible because he brought the Hindu–Arabic numeration system, including zero, to Western Europe. Algebra was referred to at that time as *Ars Magna*—The Great Art.

A third major contributor to algebra was Francois Viete (1540–1603), known as “the father of modern algebra,” who introduced the first systematic algebraic notation in his book *In Artem Analyticam*. A prominent lawyer, he also served as privy councillor to Henry IV, for whom he decoded wartime messages.
A major aspect of algebraic thinking is the concept of a variable, an understanding of which is fundamental to algebra. Whereas in basic arithmetic we have only fixed numbers, or constants, as in \( 4 + 3 = 7 \), in algebra we also have values that vary—hence the term variable. However, variable can mean several different things in mathematics.

A variable may stand for a missing element or for an unknown, as in \( x + 2 = 5 \). In this situation, while we could replace the variable in the sentence with any number, there is exactly one number that makes the sentence true. Here, when we replace the unknown \( x \) with 3, we make the statement true.

In a different situation, a variable can represent more than one thing. For example, in a group of children, you could say that their heights vary with their ages. If \( b \) represents height and \( a \) represents age, then both \( b \) and \( a \) can have different values for different children in the group. Here a variable represents a changing quantity.

Variables can also be used in generalizations of patterns, as we saw in Section 1-2. If we were to use actual values instead of variables, the instructions would only apply in a limited set of situations.

A variable can also be an element of a set, or a set itself; for example, in the definition of the intersection of two sets \( A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \), \( x \) is any element that belongs to both sets.

To apply algebra in solving problems, we frequently need to translate given information into a mathematical expression involving variables designated by letters or words. In all such examples, we may name the variables as we choose.

Variables are useful because they allow instructions to be specified in a general way. For example, if we ask each student to think of a number, double it, and add 1 to the result, these instructions may be written as \( 2x + 1 \). In mathematics, the most common letters for variables are \( x \), \( y \), and \( z \), but any other letter from the Latin or even Greek alphabet may be used. If a variable is named \( x \), then each occurrence of \( x \) in a given problem, equation, or proof refers to the same quantity.

In Examples 4-1 and 4-2, as well as in the student page that follows it, simple word statements are translated into algebraic expressions (see the student page for a definition). Notice in the student page the algebraic expressions for division. In this chapter, we will use the fact that division is the inverse of multiplication and vice versa, that is, that

***Historical Note***

Mary Everest (1832–1916), born in England and raised in France, was a self-taught mathematician and is most well known for her works on mathematics and science education. In 1855, Everest married her friend and fellow mathematician George Boole. (Mt. Everest was named after her uncle Sir George Everest.)


But when we come to the end of our arithmetic we do not content ourselves with guesses; we proceed to algebra—that is to say, to dealing logically with the fact of our own ignorance. . . .

Instead of guessing whether we are to call it nine, or seven, or a hundred and twenty, or a thousand and fifty, let us agree to call it \( x \), and let us always remember that \( x \) stands for the Unknown. . . .

This method of solving problems by honest confession of one’s ignorance is called Algebra.
Example 4-1

Write each of the following statements in algebraic form:

a. 2 more than a number
b. 2 greater than a number
c. 2 less than a number
d. 2 times a number
e. A number times itself
f. The cost of renting a car for any number of days if the charge per day is $40
g. The distance a car traveled at a constant speed of 65 mph for any number of hours

Solution

a. \( n + 2 \)
b. \( n + 2 \)
c. \( n - 2 \)
d. \( 2 \cdot n \) or \( 2n \)
e. \( n \cdot n \) or \( n^2 \)
f. If \( n \) is the number of days, the cost of renting the car for \( n \) days at $40 per day is \( 40 \cdot n \) or \( 40n \) dollars.
g. If \( b \) is the number of hours traveled at 60 mph, the total distance traveled in \( b \) hours is \( 60 \cdot b \) or \( 60b \) miles.

Example 4-2

In each of the following, translate the given information into a symbolic expression involving quantities designated by letters:

a. One weekend, a store sold twice as many CDs as full size DVDs and 25 fewer mini DVDs than CDs. If the store sold \( d \) full size DVDs, how many mini DVDs and CDs did it sell?
b. French fries have about 12 calories apiece. A hamburger has about 600 calories. Akiva is on a diet of 2000 calories per day. If he ate \( f \) french fries and one hamburger, how many more calories can he consume that day?

Solution

a. Because \( d \) full size DVDs were sold, twice as many CDs as full size DVDs implies \( 2d \) CDs. Thus, 25 fewer mini DVDs than CDs implies \( 2d - 25 \) mini DVDs.
b. First, find how many calories Akiva consumed eating \( f \) french fries and one hamburger. Then, to find how many more calories he can consume, subtract this expression from 2000.

1 french fry \( = 12 \) calories
\( f \) french fries \( = 12f \) calories

Therefore, the number of calories in \( f \) french fries and one hamburger is

\[ 600 + 12f \]

The number of calories left for the day is \( 2000 - (600 + 12f) \), or \( 2000 - 600 - 12f \), or 1400 \(- 12f \).

\[(a \div b) \cdot b = a\) and \((a \cdot b) \div b = a\) (where \( b \neq 0 \)). However, we will follow the common practice and write \( \frac{a}{b} \) for \( a \div b \); therefore \( \frac{a}{b} \cdot b = a\) and \( \frac{a \cdot b}{b} = a\) (\( b \neq 0 \)).
Variables and Expressions

How can you write an algebraic expression?

Example A

Nita bought some candles costing $4 each. How can you represent their total cost?

Make a table to show the cost for different quantities of candles. Use a letter such as $n$ to represent the number of candles. Because $n$ represents a quantity whose value can vary, it is called a variable.

The total cost of the candles is represented by $4 \times n$ or $4n$.

An algebraic expression is a mathematical expression containing variables, numbers, and operation symbols. Before you write an algebraic expression, identify the operation. The table below shows how two or more word phrases can refer to an operation.

<table>
<thead>
<tr>
<th>Word Phrase</th>
<th>Operation</th>
<th>Algebraic Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>the sum of 9 and a number $n$</td>
<td>Addition</td>
<td>$9 + n$</td>
</tr>
<tr>
<td>a number $m$ increased by 8</td>
<td></td>
<td>$m + 8$</td>
</tr>
<tr>
<td>six more than a number $t$</td>
<td></td>
<td>$t + 6$</td>
</tr>
<tr>
<td>add eighteen to a number $h$</td>
<td></td>
<td>$h + 18$</td>
</tr>
<tr>
<td>seventy-seven plus a number $r$</td>
<td></td>
<td>$77 + r$</td>
</tr>
<tr>
<td>the difference of 12 and a number $n$</td>
<td>Subtraction</td>
<td>$12 - n$</td>
</tr>
<tr>
<td>seven less than a number $y$</td>
<td></td>
<td>$y - 7$</td>
</tr>
<tr>
<td>ten decreased by a number $p$</td>
<td></td>
<td>$10 - p$</td>
</tr>
<tr>
<td>the product of 4 and a number $k$</td>
<td>Multiplication</td>
<td>$4k$</td>
</tr>
<tr>
<td>fifteen times a number $t$</td>
<td></td>
<td>$15t$</td>
</tr>
<tr>
<td>two multiplied by a number $m$</td>
<td></td>
<td>$2m$</td>
</tr>
<tr>
<td>the quotient of a number divided by five</td>
<td>Division</td>
<td>$\frac{n}{5}$</td>
</tr>
<tr>
<td>twenty-five divided by a number $m$</td>
<td></td>
<td>$\frac{25}{m}$</td>
</tr>
</tbody>
</table>

A teacher instructed her class as follows:

Take any number and add 15 to it. Now multiply that sum by 4. Next subtract 8 and divide the difference by 4. Now subtract 12 from the quotient and tell me the answer. I will tell you the original number.

Analyze the instructions to see how the teacher was able to determine the original number.

**Solution**  Translate the information into an algebraic form.

<table>
<thead>
<tr>
<th>Instructions</th>
<th>Discussion</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Take any number.</td>
<td>Since any number is used, we need a variable to represent the number. Let $n$ be that variable.</td>
<td>$n$</td>
</tr>
<tr>
<td>Add 15 to it.</td>
<td>We are told to add 15 to “it.” “It” refers to the variable $n$.</td>
<td>$n + 15$</td>
</tr>
<tr>
<td>Multiply that sum by 4.</td>
<td>We are told to multiply “that sum” by 4. “That sum” is $n + 15$.</td>
<td>$4(n + 15)$</td>
</tr>
<tr>
<td>Subtract 8.</td>
<td>We are told to subtract 8 from the product. The difference is $4(n + 15) - 8$.</td>
<td>$4(n + 15) - 8$</td>
</tr>
<tr>
<td>Divide the difference by 4.</td>
<td>Divide it by 4.</td>
<td>$\frac{4(n + 15) - 8}{4}$</td>
</tr>
<tr>
<td>Subtract 12 from the quotient and tell me the answer.</td>
<td>We are told to subtract 12 from the quotient.</td>
<td>$\frac{4(n + 15) - 8}{4} - 12$</td>
</tr>
</tbody>
</table>

Translating what the teacher told the class to do results in the algebraic expression $\frac{4(n + 15) - 8}{4} - 12$. We are also told that we have to tell the teacher the answer obtained and she then produces the original number. Let’s use the strategy of working backward to see if we can determine what happens. Suppose we tell the teacher that our final result is $r$. Think about how $r$ was obtained. Just before we told the teacher “$r$,” we had subtracted 12. To reverse that operation, we could add 12 to obtain $r + 12$. Prior to that we had divided by 4. To reverse that, we could multiply by 4 to obtain $4r + 48$. To get that result, we had subtracted 8, so that now we add 8 to obtain $4r + 56$. Just previous to that we had multiplied by 4, so now we divide $4r + 56$ by 4 to obtain $r + 14$. The first operation had been to add 15, so now we subtract 15 from $r + 14$ to get $r - 1$. Thus, the teacher knows when we tell her that our final result is $r$, it is 1 more than the number with which we started, or the number with which we started, $n$, is the result minus 1.

This can be shown as follows:

$$\frac{4(n + 15) - 8}{4} - 12 = \frac{4(n + 15 - 2)}{4} - 12$$

$$= (n + 13) - 12$$

$$= n + 1$$
Figure 4-1 shows a sequence of figures containing small square tiles. Some of the tiles are shaded. Notice that the first figure has one shaded tile. The second figure has $2 \cdot 2$, or $2^2$, shaded tiles. The third figure has $3 \cdot 3$, or $3^2$, shaded tiles. Answer the following:

a. How many shaded tiles are there in the $n$th figure?

b. How many white tiles are there in the $n$th figure?

**Solution**

a. The squares of shaded tiles have sides with increasing lengths 1, 2, 3, and so on. In the $n$th figure, the length of a side of the shaded region would be $n$. Hence, the $n$th figure has $n^2$ shaded tiles.

b. One way to think about the number of white tiles is to recognize that the number of white tiles on a side is 2 more than $n$, or $n + 2$. The number of white tiles could be 4 times $(n + 2)$, less any overlapping counting. In this case, each corner tile would be duplicated so 4 white tiles are overcounted giving us $4(n + 2) - 4$, or $4n + 4$, white tiles.

Another way to count the white tiles in the $n$th figure is to count the total number of tiles and then subtract from this total the number of shaded tiles. We have seen that the number of white tiles on the bottom side of the $n$th square is $n + 2$, and the number of the shaded tiles on a side is $n$. Thus, the number of white tiles is $(n + 2)^2 - n^2$. It can be shown that this answer is the same as $4n + 4$ obtained earlier.

**NOW TRY THIS 4-1**

a. There is another way to count the white tiles in Example 4-4. First, remove the four white corner tiles and then count the number of remaining white tiles. Complete this approach.

b. Noah has some white square tiles and some blue square tiles. They are all the same size. He first makes a row of white tiles and then surrounds the white tiles with a single layer of blue tiles, as shown in Figure 4-2.

**How many gray tiles does he need:**

i. to surround a row of 100 white tiles?

ii. to surround a row of $n$ white tiles?

Variables are commonly used in spreadsheets. To compute the 50th term of the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, …, in which the first two terms are 1, 1 and each subsequent
Algebraic Thinking

term is the sum of the two preceding terms, could take a very long time by hand or even by calculator. However, using a spreadsheet, any desired term of the Fibonacci sequence and the previous term appear instantaneously. The student page above shows how to create the Fibonacci sequence on a spreadsheet using two variables, \(A1\) and \(A2\).

At a local farmer’s market, three purchases were made for the prices shown in Figure 4-3. What is the cost of each object?

Example 4-5

At a local farmer’s market, three purchases were made for the prices shown in Figure 4-3. What is the cost of each object?
Solution  Approaches to this problem may vary. For example, if the objects in the first two purchases are put together, the total cost would be \( $8 + $9 \), or $17. That cost would be for two vases and one each of the cantaloupe and watermelon, as in Figure 4-4.

Now if the cantaloupe and watermelon are taken away from that total, then according to the cost of those two objects from the tag on the right, the cost should be reduced to $10 for two vases. That means each of the two vases costs $5. This in turn tells us that the cantaloupe costs $8 - $5, or $3, and the watermelon costs $9 - $5, or $4.

The solution in Example 4-5 could involve the strategy of writing an equation. But first we need a basic knowledge of solving equations.

Assessment 4-1A

1. Write each of the following statements in algebraic form:
   a. The third term of an arithmetic sequence whose first term is 10 and whose difference is \( d \)
   b. 10 less than twice a number
   c. 10 times the square of a number
   d. The difference between the square of a number and twice the number
2. a. Translate the following information into an algebraic form: Take any number, add 3 to it, multiply the sum by 7, subtract 14, and divide the difference by 7. Finally, subtract the original number.
   b. Simplify your answer in part (a).
3. In the tile pattern in the sequence of figures shown each figure starting from the second has two more blue squares than the preceding one. Answer the following:
   ![Figure 4-4](image)

   a. How many shaded tiles are there in the \( n \)th figure?
   b. How many white tiles are there in the \( n \)th figure?
   c. How many blue tiles are there in the \( n \)th figure?
4. In the following, write an expression in terms of the given variable that represents the indicated quantity. For example, the distance traveled at a constant speed of \( 60 \) mph during \( t \) hr could be written as \( 60t \) miles.
   a. The cost of having a plumber spend \( b \) hr at your house if the plumber charges $20 for coming to the house and $25 per hour for labor
   b. The amount of money in cents in a jar containing \( d \) dimes and some nickels and quarters if there are 3 times as many nickels as dimes and twice as many quarters as nickels
   c. The sum of three consecutive integers if the least integer is \( x \)
   d. The amount of bacteria after \( n \) min if the initial amount of bacteria is \( q \) and the amount of bacteria doubles every minute. (Hint: The answer should contain \( q \) as well as \( n \).)
   e. The temperature after \( t \) hr if the initial temperature is \( 40°F \) and each hour it drops by \( 3°F \)
   f. Pawel's total earning after 3 yr if the first year his salary was \( s \) dollars, the second year it was \$5000 higher, and the third year it was twice as much as the second year
g. The sum of three consecutive odd natural numbers if the least is $x$

h. The sum of three consecutive natural numbers if the middle is $m$

5. If the number of professors in a college is $P$ and the number of students $S$, and there are 20 times as many students as professors, write an algebraic equation that shows this relationship.

6. If $g$ is the number of girls in a class and $b$ the number of boys and if there are five more girls ($g$) than boys ($b$) in a class, write an algebraic equation that shows this relationship.

7. Ryan is building matchstick square sequences as shown. How many matchsticks will he use for the $n$th figure?

8. Write an algebraic equation relating the variables described in each of the following situations:
   a. The pay, $P$, for $t$ hr if you are paid $8$ an hour
   b. The pay, $P$, for $t$ hr if you are paid $15$ for the first hour and $10$ for each additional hour

9. For a particular event, a student pays $5$ per ticket and a nonstudent pays $13$ per ticket. If $x$ students and 100 nonstudents buy tickets, find the total revenue from the sale of the tickets in terms of $x$.

10. Suppose a will decreed that three siblings will each receive a cash inheritance according to the following: The eldest receives 3 times as much as the youngest, and twice as much as the middle sibling. Answer the following:
    a. If the youngest sibling receives $x$, how much do the other two receive in terms of $x$?
    b. If the middle sibling receives $y$, how much do the other two receive in terms of $y$?
    c. If the oldest sibling receives $z$, how much do the other two receive in terms of $z$?

---

1. Write each of the following statements in algebraic form:
   a. 10 more than a number
   b. 10 less than a number
   c. 10 times a number
   d. The sum of a number and 10
   e. The difference between the square of a number and the number

2. Translate the following into algebraic form:
   a. Take any number, add 25 to it, multiply the sum by 3, subtract 60, and divide the difference by 3. Finally, add 5.
   b. Simplify your answer in part (a).

3. Discover a possible tile pattern in the following sequence and answer the following:
   First, Second, and Third... 
   a. How many shaded tiles are there in the $n$th figure of your pattern?
   b. How many white tiles are there in the $n$th figure of your pattern?

4. If a school has $w$ women and $m$ men and you know that there are 100 more men than women, write an algebraic equation relating $w$ and $m$.

5. Suppose there are 15 more chairs ($c$) and tables ($t$) in a classroom and there are 15 more chairs than tables. Write an algebraic equation relating $c$ and $t$.

6. In the following, write an expression in terms of the given variables that represents the indicated quantity:
   a. The cost of having a plumber spend $b$ hr at your house if the plumber charges $30$ for coming to the house and $x$ per hour for labor
   b. The amount of money in cents in a jar containing some nickels and $d$ dimes and some quarters if there are 4 times as many nickels as dimes and twice as many quarters as nickels
   c. The sum of three consecutive integers if the greatest integer is $x$
   d. The amount of bacteria after $n$ min if the initial amount of bacteria is $q$ and the amount of bacteria triples every 30 sec. (Hint: The answer should contain $q$ as well as $n$.)
   e. The temperature $t$ hr ago if the present temperature is 40°F and each hour it drops by 3°F
   f. Pawel’s total earnings after 3 yr if the first year his salary was $s$ dollars, the second year it was $5000$ higher, and the third year it was twice as much as the first year
   g. The sum of three consecutive even whole numbers if the greatest is $x$
7. Ryan is building matchstick square sequences so that one square is added to the right each time, as shown. How many matchsticks will he use for the $n$th figure and for the figure one before the $n$th?

![Matchstick square sequence]

8. Write an algebraic equation relating the variables described in each of the following situations:
   a. The pay, $P$, for $t$ hr if you are paid $d$ an hour
   b. The pay, $P$, for $t$ hr if you are paid $15$ for the first hour and $6$ for each additional hour
   c. The total pay, $P$, for a visit and $t$ hr of gardening if you are paid $20$ for the visit and $10$ for each hour of gardening

Mathematical Connections 4-1

Communication

1. Students were asked to write an algebraic expression for the sum of three consecutive natural numbers. One student wrote \( x + (x + 1) + (x + 2) = 3x + 3 \). Another wrote \( (x - 1) + x + (x + 1) = 3x \). Explain who is correct and why.

Open-Ended

2. A teacher instructed her class as follows: Take any odd number, multiply it by 4, add 16, and divide the result by 2. Subtract 7 from the quotient and tell me your answer. I will tell you the original number. Explain how the teacher was able to tell each student's original number.

3. A student writes \( A \cap B = B \cap A \) and \( A \cup B = B \cup A \) are algebraic generalizations of set properties in a way similar to the statements \( a + b = b + a \) and \( ab = ba \) are generalizations of arithmetic properties of numbers. How do you respond?

4. A student claims that the sum of five consecutive integers is equal to 5 times the middle integer and would like to know if this is always true, and if so, why. He would like to know if the statement generalizes to the sum of five consecutive terms in any arithmetic sequence. How do you respond?

5. Matt has twice as many stickers as David. If David has 10 stickers, how many stickers does Matt have in terms of $d$?

6. A student writes \( a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c) \). How do you respond?

7. A student wonders if sets can ever be considered as variables. What do you tell her?

8. A student thinks that if $A$ and $B$ are sets, then the statements $A \cup B = B \cup A$ and $A \cap B = B \cap A$ are algebraic generalizations of set properties in a way similar to the statements $a + b = b + a$ and $ab = ba$ are generalizations of arithmetic properties of numbers. How do you respond?

Third International Mathematics and Science Study (TIMSS) Question

- \( [ \square ] \) represents the number of magazines that Lina reads each week. Which of these represents the total number of magazines that Lina reads in 6 weeks?
  a. \( 6 + [\square] \)
  b. \( 6 \times [\square] \)
  c. \( [\square] + 6 \)
  d. \( ([\square] + [\square]) \times 6 \)

TIMSS 2003, Grade 4

National Assessment of Educational Progress (NAEP) Question

- N stands for the number of hours of sleep Ken gets each night. Which of the following represents the number of hours of sleep Ken gets in 1 week?
  a. \( N + 7 \)
  b. \( N - 7 \)
  c. \( N \times 7 \)
  d. \( N \div 7 \)

NAEP, Grade 4, 2005
Variables are frequently associated with equations. When variables are thought of as unknowns, we can consider expressions such as \( w + c = 7 \). The equal sign indicates that the values on both sides of the equation are the same even though they do not look the same. As pointed out in the Research Note, students sometimes mistakenly think of the equal sign only as a symbol for separation.

To solve equations, we need several properties of equality. Children discover many of these by using a balance scale. For example, consider two weights of amounts \( a \) and \( b \) on the balances, as in Figure 4-5(a). If the balance is level, then \( a = b \). When we add an equal amount of weight, \( c \), to both sides, the balance is still level, as in Figure 4-5(b).

![Figure 4-5](image)

This demonstrates that if \( a = b \), then \( a + c = b + c \), which is the **addition property of equality**.

Similarly, if the scale is balanced with amounts \( a \) and \( b \), as in Figure 4-6(a), and we put additional \( a \)'s on one side and an equal number of \( b \)'s on the other side, the scale remains level, as in Figure 4-6(b).

![Figure 4-6](image)

Figure 4-6 suggests that if \( c \) is any natural number and \( a = b \), then \( ac = bc \), which is the **multiplication property of equality**. These properties are summarized next. They are true for all numbers, but in this chapter we use them just for whole numbers.

**Remark**

a. In algebra it is common to omit the multiplication sign in a product when letters are involved. Thus, we write \( ac \) instead of \( a \cdot c \) and \( 3x \) instead of \( 3 \cdot x \)

b. Under certain conditions the properties listed here can be proved and therefore are called theorems.
The properties imply that we may add the same number to both sides of an equation or multiply both sides of the equation by the same number without affecting the equality. A new statement results from reversing the order of the if and then parts of the addition property of equality. The new statement is the converse of the original statement. In the case of the addition property, the converse is a true statement. The converse of the multiplication property of equality is also true when \( c \neq 0 \). These properties are summarized next.

**Theorem 4–1: The Addition Property of Equality**

For any numbers \( a, b, \) and \( c \), if \( a = b \), then \( a + c = b + c \).

**The Multiplication Property of Equality**

For any numbers \( a, b, \) and \( c \), if \( a = b \), then \( ac = bc \).

Equality is not affected if we substitute a number for its equal. This property is referred to as the substitution property. Examples of substitution follow:

1. If \( a + b = c + d \) and \( d = 5 \), then \( a + b = c + 5 \).
2. If \( a + b = c + d \), \( b = e \), and \( d = f \), then \( a + e = c + f \).

Using the substitution property we can see that equations can be added or subtracted “side by side”; that is, we have the following:

**Theorem 4–3: Addition and Subtraction Property of Equations**

If \( a = b \) and \( c = d \), then \( a + c = b + d \) and \( a - c = b - d \).

This property can be justified as follows: Using the addition property of equality, if \( a = b \), then \( a + c = b + c \). Substituting \( d \) for \( c \) on the right side, we get \( a + c = b + d \). Similarly, \( a - c = b - d \).

Widely used properties in early grades are the commutative and associative properties of addition and multiplication that can be performed in any order, for example, \( 2 + 8 = 8 + 2 \) and \( 2 \cdot 8 = 8 \cdot 2 \). Also, \( 2 + (8 + 5) = (2 + 8) + 5 \), and in general, we have the following:

**Theorem 4–4: Commutative Properties of Addition and Multiplication**

\[ a + b = b + a, \quad ab = ba \]

**Theorem 4–5: Associative Properties of Addition and Multiplication**

\[ (a + b) + c = a + (b + c), \quad (ab)c = a(bc) \]

On the student page that follows you will see two of the properties just discussed.
**Warm Up**

1. \(4 \times 8\)
2. \(6 \times 8\)
3. \(7 \times 7\)
4. \(7 \times 8\)
5. \(6 \times 9\)
6. \(7 \times 9\)

**What’s the pattern?**

a. Use a calculator to find each product.

\[
\begin{align*}
3 \times 5 & = 15 \\
3 \times 50 & = 150 \\
3 \times 500 & = 1500 \\
30 \times 5 & = 150 \\
30 \times 50 & = 1500 \\
30 \times 500 & = 15000 \\
300 \times 5 & = 1500 \\
300 \times 50 & = 15000 \\
300 \times 500 & = 150000
\end{align*}
\]

b. Find the following products without a calculator. Then check your answers with a calculator.

\[
5 \times 8, \ 50 \times 8, \ 50 \times 80, \ 500 \times 8, \ 500 \times 80, \ 500 \times 800
\]

c. Describe a rule that tells how to find each product.

**How can properties help you multiply more easily?**

**Commutative Property of Multiplication**

You can change the order of the factors.

\[
34 \times 8 = 8 \times 34
\]

**Associative Property of Multiplication**

You can change the grouping of factors.

\[
(7 \times 25) \times 4 = 7 \times (25 \times 4)
\]

**Example A**

Find \(20 \times 5 \times 6\).

Using the Associative Property, you can think:

\[
(20 \times 5) \times 6 = 100 \times 6 = 600 \text{ OR } 20 \times (5 \times 6) = 20 \times 30 = 600.
\]

**Example B**

Find \(2 \times 70 \times 50\).

Use the properties to change the order and the groupings.

\[
2 \times 70 \times 50 = 2 \times (70 \times 50) = 2 \times (50 \times 70) = (2 \times 50) \times 70 = (2 \times 70) \times 50 = 100 \times 70 = 7000.
\]

**Talk About It**

1. How is the Associative Property used in Example B?
2. How is the Commutative Property used in Example B?
3. Can you use the Associative Property for \(2 \times (5 + 6)\)? Explain.

Theorem 4–6: The Distributive Property of Multiplication over Addition
For all numbers \(a, b,\) and \(c\), \(a(b + c) = ab + ac\).

Notice that we used this property in the solution of Example 4–3, where we wrote \(4(n + 15)\) as \(4n + 4 \cdot 15\). Also notice the analogous property among sets: \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\). Similarly, we have

Theorem 4–7: The Distributive Property of Multiplication over Subtraction
For all numbers \(a, b,\) and \(c\), \(a(b - c) = ab - ac\).

Notice that using the commutative property of multiplication, each of the distributive properties above can be written in the equivalent forms

\[(b + c)a = ba + ca,\] and
\[(b - c)a = ba - ca\]

When the distributive properties are written from right to left, we refer to them as factoring. Thus, \(ab + ac = a(b + c)\) and \(ab - ac = a(b - c)\). We say that \(a\) has been “factored out.”

Solving Equations
Part of algebraic thinking involves operations on numbers and other elements represented by symbols. Finding solutions to equations is a major part of algebra. As the Research Note points out, use of tangible objects can increase students’ engagement and comprehension when they work with equations. The balance-scale model is an excellent way to foster understanding of the basic concepts used in solving equations and inequalities, and equations can be explored using a balance scale. Inequalities tilt the scale.
Algebraic Thinking

For example, consider Figure 4-7. If we release the pan on the left, what will happen? Upon release, the scale will tilt down on the right side and we have an inequality, $2 \cdot 3 < 3 + (2 \cdot 2)$.

![Figure 4-7](image)

Next consider Figure 4-8. If we release the pan, then the sides will balance and we have the equality $2 \cdot 3 = (1 + 1) + 4$.

![Figure 4-8](image)

A balance scale can also be used to reinforce the idea of a replacement set for a variable. Name some solutions in Figure 4-9 that will keep the scale balanced. For example, $3 \cdot 2$ balances $2 \cdot 3$, $3 \cdot 6$ balances $2 \cdot 9$, and so on. Do you see any patterns in the numbers that will balance the scale?

![Figure 4-9](image)

Other types of balance-scale problems can also help students get ready for algebra. Work through Now Try This 4-3 before proceeding.
To solve equations, we may use the properties of equality developed earlier. Consider $3x - 14 = 1$. Put the equal expressions on the opposite pans of the balance scale. Because the expressions are equal, the pans should be level, as in Figure 4-11.

\[
\begin{align*}
3x &- 14 = 1 \\
\end{align*}
\]

To solve for $x$, we use the properties of equality to manipulate the expressions on the scale so that after each step, the scale remains level and, at the final step, only an $x$ remains on one side of the scale. The number on the other side of the scale represents the solution to the original equation. To find $x$ in the equation of Figure 4-11, consider the scales pictured in successive steps in Figure 4-12. In Figure 4-12 each successive scale represents an equation that is equivalent to the original equation; that is, each has the same solution as the original. The last scale shows $x = 5$. To check that 5 is the correct solution, we substitute 5 for $x$ in the original equation. Because $3 \cdot 5 - 14 = 1$ is a true statement, 5 is the solution to the original equation.

\[
\begin{align*}
3x - 14 &= 1 \\
\end{align*}
\]
NOW TRY THIS 4-4 Notice the use of concrete objects in solving equations on the student page and answer the “Talk About It” question that follows the balance-scale model.

Example 4-6

Solve each of the following for $x$:

a. $x + 4 = 20$

b. $3x = x + 10$

c. $4x + 5x = 99$

d. $4(x + 3) + 5(x + 3) = 99$

Solution

a. $x + 4 = 20$ implies $(x + 4) - 4 = 20 - 4; x = 16.$

b. $3x = x + 10$ implies $3x - x = 10 + x - x; 3x - 1x = 10; (3 - 1)x = 10; 2 = 10; x = 5.$

c. $4x + 5x = 99; (4 + 5)x = 99; 9x = 99; x = 11$

d. We could multiply out and get

$$4(x + 3) + 5(x + 3) = 99; 4x + 12 + 5x + 15 = 99$$

$$9x + 27 = 99; 9x + 27 - 27 = 99 - 27; 9x = 72, x = 8$$

Or we could have thought of $x + 3$ as a new unknown $\square$. So if $x + 3 = \square$, we get $4\cdot\square + 5\cdot\square = 99$, so $9\cdot\square = 99, \square = 11$, which implies $x + 3 = 11$ and hence $x = 8$.

Historical Note

Born into a well-to-do family in Scotland, Mary Fairfax (1780–1872) first studied simple arithmetic only briefly at the age of 13. Also at about this time she came upon some mysterious symbols in a women’s fashion magazine and, after persuading her brother’s tutor to purchase some elementary literature for her on the subject, began her study of algebra. Later as a young mother and widow, she obtained a small library of works to provide her with a sound background in mathematics. Throughout the rest of her life, Somerville distinguished herself as a skilled scientific writer respected among her colleagues, publishing a number of works. Her last scientific book, Molecular and Microscopic Science, was published in 1869 when she was 89. In her autobiography Mary wrote of how she “was sometimes annoyed when in the midst of a difficult problem” a visitor would enter. Shortly before her death she wrote

I am now in my ninety-second year, . . . , I am extremely deaf, and my memory of ordinary events, and especially of the names of people, is failing, but not for mathematical and scientific subjects. I am still able to read books on the higher algebra for four or five hours in the morning and even to solve the problems. Sometimes I find them difficult, but my old obstinacy remains, for if I do not succeed today, I attack them again tomorrow.

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**Solving Equations with Whole Numbers**

How can you solve an equation?

When you solve an equation, you find the value of the variable that makes the equation true.

**Example A**

Wynn sold 6 sketches, each for the same amount, and made $180 in sales. How much did he charge for each sketch?

Let \( s \) equal the amount for each sketch.

Then the equation is \( 6s = 180 \).

### What You Write

<table>
<thead>
<tr>
<th></th>
<th>Balancing the Pans</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 6s = 180 )</td>
<td>The pans are balanced.</td>
</tr>
<tr>
<td>( 6s + 6 = 180 + 6 )</td>
<td>180 has been separated into 6 equal parts.</td>
</tr>
<tr>
<td>( s = 30 )</td>
<td>Each $5 equals $5</td>
</tr>
</tbody>
</table>

Wynn charged $30 for each sketch.

**Talk About It**

1. Why was each side of the equation in Example A divided by 6?

**How can you check your answer?**

To check your answer, substitute it for the variable in the original equation. In Example A, substitute 30 for \( s \) in \( 6s = 180 \).

**Check:**

\[
\begin{align*}
6s &= 180 \\
6(30) &= 180 \\
180 &= 180
\end{align*}
\]

When both sides of the equation can be simplified to the same number, the value of the variable is correct.


**Application Problems**

The simple model in Figure 4-13 demonstrates a method for solving application problems. Formulate the problem with a mathematical model, solve that mathematical model, and then interpret the solution in terms of the original problem.

![Figure 4-13](Image)

An example of this model at the third-grade level appears in Figure 4-14.

![Figure 4-14](Image)

NOW TRY THIS 4-5 Read the following student page from a fourth-grade textbook and answer the “Talk About It” questions.

We can apply Pólya’s four-step problem-solving process to solving word problems in which the use of algebraic thinking is appropriate. In Understanding the Problem, we identify what is given and what is to be found. In Devising a Plan, we assign letters to the unknown quantities and try to translate the information in the problem into a model involving equations. In Carrying Out the Plan, we solve the equations or inequalities. In Looking Back, we interpret and check the solution in terms of the original problem.

In the following problems, we demonstrate Pólya’s four-step problem-solving process.
Translating Words to Equations

LEARN

How do you write an equation?

A Great Dane puppy weighed 4 pounds when it was born. After 3 weeks, it weighed 6 pounds.

Example

Write an equation to show how much weight the puppy gained in 3 weeks.

\[
4 \text{ lb} + p = 6 \text{ lb}
\]

Weight at birth + Pounds gained = Weight at 3 weeks

4 + p = 6

The equation 4 + p = 6 shows how much weight the puppy gained in 3 weeks.

Talk About It

1. What does \( p \) stand for in the Example?
2. After 5 weeks, the puppy weighed 8 pounds. Write an equation to show how much weight the puppy gained in 5 weeks.

CHECK

Write an equation for each sentence.

1. \( p \) pages plus 7 pages equals 17 pages.
2. 8 less than \( k \) is 15.
3. 9 times \( n \) is 27.
4. 36 divided by \( y \) is 12.
5. **Reasoning** Kate wanted to find how many centimeters equal 80 millimeters. She used the equation \( 10x = 80 \). Is Kate’s equation correct? Explain.
**Problem Solving** Overdue Books

Bruno has five books overdue at the library. The fine for overdue books is 10¢ a day per book. He remembers that he checked out an astronomy book a week before he checked out four novels. If his total fine was $8.70, how long was each book overdue?

**Understanding the Problem** Bruno has five books overdue. He checked out an astronomy book seven days earlier than the four novels, so the astronomy book is overdue seven days more than the novels. The fine per day for each book is 10¢, and the total fine was $8.70. We need to find out how many days each book is overdue.

**Devising a Plan** Let \( d \) be the number of days that each of the four novels is overdue. The astronomy book is overdue seven days longer, that is, \( d + 7 \) days. To write an equation for \( d \), we express the total fine in two ways. The total fine is $8.70. This fine in cents equals the fine for the astronomy book plus the fine for the four novels.

\[
\text{Fine for each of the novels} = \frac{\text{Fine per day times number of overdue days}}{10} \cdot \frac{d}{d}
\]

\[
\text{Fine for the four novels} = \frac{1}{4} \text{ day's fine for novels times number of overdue days} \cdot \frac{10}{d}
\]

\[
= (4 \cdot 10)d
\]

\[
= 40d
\]

**Devising a Plan** Let \( d \) be the number of days that each of the four novels is overdue. The astronomy book is overdue seven days longer, that is, \( d + 7 \) days. To write an equation for \( d \), we express the total fine in two ways. The total fine is $8.70. This fine in cents equals the fine for the astronomy book plus the fine for the four novels.

\[
\text{Fine for the four novels} + \text{Fine for the astronomy book} = \text{Total fine}
\]

\[
40d + 10(d + 7) = 870
\]

\[
40d + 10d + 70 = 870
\]

\[
50d + 70 = 870
\]

\[
50d = 870 - 70
\]

\[
50d = 800
\]

\[
d = 16
\]

Thus, each of the four novels was 16 days overdue, and the astronomy book was overdue \( d + 7 \), or 23, days.

**Looking Back** To check the answer, follow the original information. Each of the four novels was 16 days overdue, and the astronomy book was 23 days overdue. Because the fine was 10¢ per day per book, the fine for each of the novels was \( 16 \times 10¢ \), or 160¢. Hence, the fine for all four novels was \( 4 \times 160¢ \), or 640¢. The fine for the astronomy book was \( 23 \times 10¢ \), or 230¢.
Consequently, the total fine was \( 230\epsilon \). Consequently, the total fine was \( 640\epsilon + 230\epsilon \), or \( 870\epsilon \), which agrees with the given information of \$8.70 as the total fine.

The problem can also be solved without algebra. One way is to notice that the astronomy book was overdue for 7 days for a fine of \( 70\epsilon \) before the other four books were overdue. Thus, \( 870\epsilon - 70\epsilon \), or \( 800\epsilon \), is the fine for the remaining five books. Therefore, the fine for one book is \( 800\epsilon / 5 \), or \( 160\epsilon \). Because the fine is \( 10\epsilon \) per day, each book was overdue \( 160/10 \), or 16, days. The astronomy book was checked out a week earlier and hence was overdue 7 days longer for 23 days.

---

**Problem Solving**  **Newspaper Delivery**

In a small town, three children deliver all the newspapers. Abby delivers 3 times as many papers as Bob, and Connie delivers 13 more than Abby. If the three children delivered a total of 496 papers, how many papers does each deliver?

**Understanding the Problem**  The problem asks for the number of papers that each child delivers. It gives information that compares the number of papers that each child delivers as well as the total number of papers delivered in the town.

**Devising a Plan**  Let \( a \), \( b \), and \( c \) be the number of papers delivered by Abby, Bob, and Connie, respectively. We translate the given information into algebraic equations as follows:

- Abby delivers 3 times as many papers as Bob: \( a = 3b \)
- Connie delivers 13 more papers than Abby: \( c = a + 13 \)
- Total delivery is 496: \( a + b + c = 496 \)

To reduce the number of variables, substitute \( 3b \) for \( a \) in the second and third equations:

\[
\begin{align*}
    c &= a + 13 \quad \text{becomes} \quad c = 3b + 13 \\
    a + b + c &= 496 \quad \text{becomes} \quad 3b + b + c = 496
\end{align*}
\]

Next, make an equation in one variable, \( b \), by substituting \( 3b + 13 \) for \( c \) in the equation \( 3b + b + c = 496 \), solve for \( b \), and then find \( a \) and \( c \).

**Carrying Out the Plan**

\[
\begin{align*}
    3b + b + 3b + 13 &= 496 \\
    7b + 13 &= 496 \\
    7b &= 483 \\
    b &= 69
\end{align*}
\]

Thus, \( a = 3b = 3 \times 69 = 207 \). Also, \( c = a + 13 = 207 + 13 = 220 \). So, Abby delivers 207 papers, Bob delivers 69 papers, and Connie delivers 220 papers.

**Looking Back**  To check the answers, follow the original information, using \( a = 207 \), \( b = 69 \), and \( c = 220 \). The information in the first sentence, “Abby delivers 3 times as many papers as Bob,” checks, since \( 207 = 3 \times 69 \). The second sentence, “Connie delivers 13 more papers than Abby,” is true because \( 220 = 207 + 13 \). The information on the total delivery checks, since \( 207 + 69 + 220 = 496 \).
NOW TRY THIS 4-6

1. Solve the Newspaper Delivery problem above by introducing only one unknown for the number of newspapers Bob delivered.
2. Recall that Example 4-5 was solved without using equations, and following the solution we mentioned that the problem can be solved using the strategy of writing an equation. You are ready now to approach the problem in this way. Suppose the price of the cantaloupe in dollars is \( x \), the price of the flower and the vase \( y \), and the watermelon \( z \). Check that from the information in Figure 4-3(a) we get \( x + y = 8 \), and write the equations corresponding to parts (b) and (c) of the figure. Solve the equations by reducing them to two equations with two unknowns and then one equation with one unknown.

Assessment 4-2A

1. Solve each of the following, if possible:
   - a. \( x - 3 = 21 \)
   - b. \( 2x + 5 = x + 25 \)
   - c. \( 2x + 5 = 3x - 4 \)
   - d. \( 5(2x + 1) + 7(2x + 1) = 84 \)
   - e. \( 3(2x - 6) = 4(2x - 6) \)

   Solve problems 3 through 10 by setting up and solving an equation.

2. Ryan is building matchstick square sequences so that one square is added to the right each time a new figure is formed, as shown. He used 67 matchsticks to form the last figure in his sequence. How many squares are in his last figure?

3. For a particular event, 812 tickets were sold for a total of $1912. If students paid $2 per ticket and nonstudents paid $3 per ticket, how many student tickets were sold?
4. An estate of $486,000 is left to three siblings. The eldest receives 3 times as much as the youngest. The middle sibling receives $14,000 more than the youngest. How much did each receive?
5. A 10 ft board is to be cut into three pieces, two equal-length ones and the third 3 in. shorter than each of the other two. If the cutting does not result in any length being lost, how long are the pieces?
6. A box contains 67 coins, only dimes and nickels. The amount of money in the box is $4.20. How many dimes and how many nickels are in the box?
7. A farmer has 700 yd of fencing to enclose a rectangular pasture for her goats. Since one side of the pasture borders a river, that side does not need to be fenced. The side parallel to the river must be twice as long as the side perpendicular to the river. Find the dimensions of the rectangular pasture.

\[ \text{River} \]
\[ \text{Pasture} \]
Assessment 4-2B

1. a. Which shape weighs the most? Tell why.
   b. Which shape weighs the least? Tell why.

2. Solve each of the following, if possible:
   a. $3x + 13 = 2x + 100$
   b. $2x + 5 = 2(x + 5)$
   c. $7(3x + 6) + 5(3x + 6) = 144$
   d. $22 - x = 3x + 6$
   e. $22 - (2x - 6) = 3(2x - 6) + 6$
   f. $5(2x - 10) = 4(2x - 10)$

   Solve problems 3 through 11 by setting up and solving an equation.

3. Ryan is building matchstick square sequences, as shown. He used 599 matchsticks to form the last two figures in his sequence. How many matchsticks did he use in each of the last two figures?

Mathematical Connections 4-2

Communication

1. Students were asked to find three consecutive whole numbers whose sum is 393. One student wrote the equation $x + (x + 1) + (x + 2) = 393$. Another wrote $(x - 1) + x + (x + 1) = 393$. Can either approach work to find the answer to the question? Explain why or why not.

2. Explain how to solve the equation $3x + 5 = 5x - 3$ using a balance scale.

Open-Ended

3. Create an equation with $x$ on both sides of the equation for each of the following:
   a. Every whole number is a solution.
   b. No whole number is a solution.
   c. 0 is a solution.

Cooperative Learning

4. Examine several elementary school textbooks for grades 1 through 5 and report on how algebraic concepts involving equations are introduced in each.

Questions from the Classroom

5. A student claims that the equation $3x = 5x$ has no solution because $3 \neq 5$. How do you respond?

6. A student claims that because in the following problem we need to find three unknown quantities, he must set up equations with three unknowns. How do you respond?

   Abby delivers twice as many papers as Jillian, and Brandy delivers 50 more papers than Abby. How many papers does each deliver if the total number of papers delivered is 550?
7. A student was told that in order to check a solution to a word problem like the one in problem 6, it is not enough to check that the solution she found satisfies the equation she set up, but rather that it is necessary to check the answer against the original problem. She would like to know why. How do you respond?

8. On a test, a student was asked to solve the equation \(4x + 5 = 3(x + 15)\). She proceeded as follows:

\[
4x + 5 = 3x + 45 = x + 5 = 45 = x = 40
\]

Hence, \(x = 40\). She checked that \(x = 40\) satisfies the original equation; however, she did not get full credit for the problem. She wants to know why. How do you respond?

Review Problems

9. If the number of sophomores, juniors, and seniors combined is denoted by \(x\) and it is 3 times the number of freshmen, denoted by \(y\), write an algebraic equation that shows the relationship.

10. Write the sum of five consecutive even numbers if the middle one is \(n\). Simplify your answer.

11. If Julie has twice as many CDs as Jack and Tira has 3 times as many as Julie, write an algebraic expression for the number of CDs each has in terms of one variable.

12. Write an algebraic equation relating the variables described in each of the following:
   a. The pay \(P\) for \(t\) hr if you are paid $30 for the first hour and $5 more than the preceding hour for each hour thereafter.

---

**Functions**

The concept of a function is central to all of mathematics and particularly to algebra, as elaborated in the following excerpt from the *Principles and Standards*:

By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school. For example, systematic experience with patterns can build up to an understanding of the idea of function (Erick Smith forthcoming), and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions. By learning that situations often can be described using mathematics, students can begin to form elementary notions of mathematical modeling. (p. 37)

Functions can model many real-world phenomena, as we shall see in this section and in later chapters. In this section, we will explore different ways to represent functions—as *rules, machines, equations, arrow diagrams, tables, ordered pairs, and graphs*. It is important that students see a variety of ways of representing functions, as indicated in the research note.
Functions as Rules

The following is an example of a game called “guess my rule,” often used to introduce the concept of a function.

When Tom said 2, Noah said 5. When Dick said 4, Noah said 7. When Mary said 10, Noah said 13. When Liz said 6, what did Noah say? What is Noah’s rule?

The answer to the first question may be 9, and the rule could be “Take the original number and add 3”; that is, for any number \( n \), Noah’s answer is \( n + 3 \).

Guess the teacher’s rule for the following responses:

<table>
<thead>
<tr>
<th>a.</th>
<th>b.</th>
<th>c.</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>Teacher</td>
<td>You</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>10</td>
</tr>
</tbody>
</table>

Solution

a. The teacher’s rule could be “Multiply the given number \( n \) by 3,” that is, \( 3n \).

b. The teacher’s rule could be “Double the original number \( n \) and add 1,”; that is, \( 2n + 1 \).

c. The teacher’s rule could be “If the number \( n \) is even, answer 0; if the number is odd, answer 1.” Another possible rule is “If the number is less than 5, answer 0; if greater than or equal to 5, answer 1.”

Functions as Machines

Another way to prepare students for the concept of a function is by using a “function machine.” The following student page shows an example of a function machine. What goes in the machine is referred to as input and what comes out as output. Thus, on the student page, if the input to the function is 2, the output is 110. Note that the output here is denoted by \( d \).

In later grades, a special notation for the output is used. For any input element \( x \), the output is denoted \( f(x) \), read “\( f \) of \( x \)”. For the function \( d = 55t \) in the example on the student page, if \( f \) is the function then when the input is 2, the output could be written as \( d(2) \). Because the output is 110, we have \( d(2) = 110 \). Since the function works according to the rule \( d(t) = 55t \), we have \( d(2) = 2 \cdot 55 = 110 \).

Historical Note

The Babylonians of Mesopotamia (ca. 2000 BCE) developed a precursor to what we today call a function. To them, a function was a table or a correspondence. Two tablets found at Senkerah on the Euphrates in 1854 give squares of the numbers up to 59 and cubes of the numbers up to 32.

In the seventeenth century the idea of a function underwent further development. In his book Geometry (1637), René Descartes (1596–1650) used the concept to describe many mathematical relationships. Almost 50 years after the publication of Descartes’ book, Gottfried Wilhelm Leibniz (1646–1716) introduced the term function. The idea of a function was further formalized by Leonhard Euler (pronounced “oiler,” 1707–1783), who introduced the notation for a function, \( y = f(x) \). Further contributions to the concept came from the mathematicians Joseph-Louis Lagrange (1736–1813) and Jean Joseph Fourier (1768–1830).

* When a function designates a specific quantity, such as \( d \) (distance) on the student page, \( d(x) \) can be used instead of \( f(x) \).
Function Rules

What You’ll Learn
To write and evaluate functions

Why Learn This?
The distance you travel in a car depends on the driving time. When one quantity depends on another, you say that one is a function of the other. So distance is a function of time. You can use functions to help you make predictions.

In the diagram at the right, an input goes through the “function machine” to produce an output.

A function is a relationship that assigns exactly one output value for each input value.

EXAMPLE Writing a Function Rule

1. Cara You are traveling in a car at an average speed of 55 mi/h. Write a function rule that describes the relationship between the time and the distance you travel.

You can make a table to solve this problem.

<table>
<thead>
<tr>
<th>Input: time (h)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output: distance (mi)</td>
<td>55</td>
<td>110</td>
<td>165</td>
<td>220</td>
</tr>
</tbody>
</table>

distance in miles = 55 \cdot \text{time in hours }

Use variables \( d \) and \( t \) for distance and time.

Quick Check

1. Write a function rule for the relationship between the time and the distance you travel at an average speed of 62 mi/h.
Consider the function machine in Figure 4-15. For the function named \( f \), what will happen if the numbers 0, 1, 3, and 6 are entered?

**Solution** If the numbers output are denoted by \( f(x) \), the corresponding values can be described using Table 4-1. Notice that in Figure 4-15 is the output from the “\( f \)” function when the input is 4.

### Example 4-8

![Figure 4-15](image)

### Functions as Equations

We can write an equation to depict the rule in Example 4-8 as follows. If the input is \( x \), the output is \( x + 3 \); that is, \( f(x) = x + 3 \). The output values can be obtained by substituting the values 0, 1, 3, 4, and 6 for \( x \) in \( f(x) = x + 3 \), as shown:

- \( f(0) = 0 + 3 = 3 \)
- \( f(1) = 1 + 3 = 4 \)
- \( f(3) = 3 + 3 = 6 \)
- \( f(4) = 4 + 3 = 7 \)
- \( f(6) = 6 + 3 = 9 \)

Notice the input and output terminology in the function machine representation on the last student page.

In many applications, both the inputs and the outputs of a function machine are numbers. However, inputs and outputs can be any objects. For example, consider a particular candy machine that accepts only 25¢, 50¢, and 75¢ and outputs one of three types of candy with costs of 25¢, 50¢, and 75¢, respectively. A function machine associates *exactly one output with each input*. If you enter some element \( x \) as input and obtain \( f(x) \) as output, then every time you enter the same \( x \) as input, you will obtain the same \( f(x) \) as output. The idea of a function machine associating exactly one output with each input according to some rule leads to the following definition.

### Definition of Function

A function from set \( A \) to set \( B \) is a correspondence from \( A \) to \( B \) in which each element of \( A \) is paired with one, and only one, element of \( B \).
The set \( A \) in the previous definition is the set of all allowable inputs and is the **domain** of the function. The set \( B \), the **codomain**, is any set that includes all the possible outputs. The set of all outputs is the **range** of the function. Set \( B \) in the definition is any set that includes the range and can be the range itself. The distinction is made for convenience sake, since sometimes the range is not easy to find. For example, consider corresponding to each student at a university the student’s identification number. This is a function from the set of all students to the set \( W \) of whole numbers. Thus the codomain is a subset of the range. The range in this case is all the identification numbers of students who are enrolled at the university. The range is a proper subset of the set \( W \). Normally, if no domain is given to describe a function, then the domain is assumed to contain all elements for which the rule is meaningful. As the Research Note points out, these concepts can be difficult for students to grasp.

A calculator contains many functions. Suppose a student enters on the calculator that has a constant key, \( 9 \times 2 + 0 \). The student then presses \( 0 \) and hands the calculator to another student. The other student is to determine the rule by entering various numbers followed by the \( \text{K} \) key. Machines with an automatic constant feature can also be used.

Other buttons on a calculator are function buttons. For example, the \( \sqrt{} \) button always displays an approximation for \( \pi \), such as 3.1415927; the \( \pm \) button either displays a negative sign in front of a number or removes an existing negative sign; and the \( x^2 \) and \( \sqrt{} \) buttons square numbers and take the square root of numbers, respectively.

Are all input-output machines function machines? Consider the machine in Figure 4-16. For any natural-number input \( x \), the machine outputs a number that is less than \( x \). If, for example, you input the number 10, the machine may output 9, since 9 is less than 10. If you input 10 again, the machine may output 3, since 3 is less than 10. Such a machine is not a function machine because the same input may give different outputs.

A high-end bicycle manufacturer incurs a daily fixed cost of $1400 for overhead expenses and a cost of $500 per bike manufactured. Answer the following:

a. Find the cost \( C(x) \) of manufacturing \( x \) bikes in a day.

b. If the manufacturer sells each bike for $700, and the profit (or loss) in producing and selling \( x \) bikes in a day is \( P(x) \), find \( P(x) \) in terms of \( x \).

c. Find the break-even point, that is, the number of bikes, \( x \), produced and sold at which break-even occurs (to break even means to make neither a profit nor a loss).

**Solution**

a. Since the cost of producing a single bike is $500, the cost of producing \( x \) bikes is 500\( x \) dollars. Because of the fixed cost of $1400 per day, the total cost, \( C(x) \) in dollars, of producing \( x \) bikes in a given day is \( C(x) = 500x + 1400 \).

b. \[ P(x) = 700x - (500x + 1400) = 200x - 1400 \]

c. We need to find the number of bikes \( x \) to be produced so that \( P(x) = 0 \); that is, we need to solve \( 200x - 1400 = 0 \). This equation is equivalent to \( 200x = 1400 \) or \( x = \frac{1400}{200} \) or 7.

Thus, the manufacturer needs to produce and sell 7 bikes to break even.
Functions as Arrow Diagrams

Arrow diagrams can be used to examine whether a correspondence represents a function. This representation is normally used when sets $A$ and $B$ are finite sets with few elements. The following example shows how arrow diagrams can be used to examine both functions and nonfunctions.

Example 4-10

Which, if any, of the parts of Figure 4-17 exhibit a function from $A$ to $B$? If a correspondence is a function from $A$ to $B$, find the range of the function.

![Figure 4-17](image)

Solution

a. Figure 4-17(a) does not define a function from $A$ to $B$, since the element 1 is paired with both 2 and 4.

b. Figure 4-17(b) does not define a function from $A$ to $B$, since the element $b$ is not paired with any element of $B$. (It is a function from a subset of $A$ to $B$.)

c. Figure 4-17(c) does define a function from $A$ to $B$, since there is one, and only one, arrow leaving each element of $A$. The fact that $d$, an element of $B$, is not paired with any element in the domain does not violate the definition. The range is $\{a, b, c\}$ and does not include $d$ because $d$ is not an output of this function, as no element of $A$ is paired with $d$.

d. Figure 4-17(d) illustrates a function, since there is one and only one arrow leaving each element in $A$. It does not matter that an element of set $B$, Brown, has two arrows pointing to it. The range is $\{\text{Brown, Smith, Doe}\}$.

e. Figure 4-17(e) illustrates a function whose range is $\{1, 4, 7, 10\}$.
Figure 4-17(e) also illustrates a one-to-one correspondence between $A$ and $B$. In fact, any one-to-one correspondence between $A$ and $B$ defines a function from $A$ to $B$ as well as a function from $B$ to $A$.

NOW TRY THIS 4-7 Determine which of the following are functions from the set of natural numbers to \{0, 1\}. Justify your reasoning.

a. For every natural-number input, the output is 0.
b. For every natural-number input, the output is 0 if the input is an even number, and the output is 1 if the input is an odd number.

Functions as Tables and Ordered Pairs

Another useful way to describe a function is with a table. Consider the information in Table 4-2 relating the amount spent on advertising and the resulting sales in a given month for a small business. Note that for Amount of Advertising and Amount of Sales, the information is actually given in thousands of dollars. We could talk about a function between the amount of dollars in Advertising and the amount of dollars in Sales, or we could simply define the function as follows: If $A = \{0, 1, 2, 3, 4\}$ and $S = \{1, 3, 5, 7, 9\}$, the table describes a function from $A$ to $S$, where $A$ represents thousands of dollars in advertising and $S$ represents thousands of dollars in sales.

<table>
<thead>
<tr>
<th>Amount of Advertising (in $1000s)</th>
<th>Amount of Sales (in $1000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

The function could be given using ordered pairs. When 0 is the input and 1 is the output, that is recorded as the ordered pair (0, 1). Similarly, the information in the second row is recorded as (1, 3) and the rest of the information as (2, 5), (3, 7), and (4, 9). The first component in the ordered pair is always an input element in the domain and the second is the corresponding output.

Example 4-11

Which of the following sets of ordered pairs represent functions? If a set represents a function, give its domain and range. If it does not, explain why.

a. $\{(1, 2), (1, 3), (2, 3), (3, 4)\}$  
b. $\{(1, 2), (2, 3), (3, 4), (4, 5)\}$  
c. $\{(1, 0), (2, 0), (3, 0), (4, 4)\}$  
d. $\{(a, b) \mid a \in N \text{ and } b = 2a\}$

Solution  

a. This is not a function because the input 1 has two different outputs.

b. This is a function with domain \{1, 2, 3, 4\}. Because the range is the possible set of outputs, the range is \{2, 3, 4, 5\}. 

NOW TRY THIS 4-7 Determine which of the following are functions from the set of natural numbers to \{0, 1\}. Justify your reasoning.
c. This is a function with domain \{1, 2, 3, 4\} and range \{0, 4\}. The output 0 appears more than once, but this does not contradict the definition of a function in that each input corresponds to only one output.

d. This is a function with domain \(\mathbb{N}\) and range \(\mathbb{E}\), the set of all even natural numbers.

**Functions as Graphs**

Perhaps one of the most widely recognized representations of a function is as a graph. Graphs, as visual representations of functions, appear in newspapers and books and on television. To graph the function created from Table 4-2, consider the set of ordered pairs\(\{(0, 1), (1, 3), (2, 5), (3, 7), (4, 9)\}\) and match each ordered pair to a point on the grid in Figure 4-18. We use the horizontal scale for the inputs and the vertical scale for the outputs and mark the point corresponding to \((0, 1)\) by starting at 0 on the horizontal scale and going up 1 unit on the vertical scale. To mark the point that corresponds to \((1, 3)\), we start at 0 and move 1 unit horizontally and then 3 units vertically. Marking the point that corresponds to an ordered pair is referred to as **graphing** the ordered pair. The set of all points that correspond to all the ordered pairs is the graph of the function. Notice that the graph consists of five points. The points are connected by a dashed line to emphasize that they lie on a straight line, but not every point on the line belongs to the graph.

Notice Examples A and C on the two partial student pages that follow. The graph in Example C consists of points that lie on a straight line. Why may we connect the points by a dashed line rather than a solid line?
School Book Page  RULES, TABLES, AND GRAPHS

Lesson 3-15  
Algebra  
**Key Idea**  
Rules, tables, and graphs can be used to show how one quantity is related to another.

Vocabulary  
- variable (p. 100)  
- table of values

Materials  
- grid paper or graph paper

**WARM UP**  
Evaluate each expression for $x = 3$.

1. $x + 8$  
2. $2x$  
3. $17 - x$  
4. $\frac{6}{2}$  
5. $7x + 8$  
6. $\frac{6}{2} - 2$

**LEARN**  
How do you use a rule to make a table?

Tickets to the county fair are $3 per person plus $1 for parking. A rule can be written to show that the total cost is $3 times the number of people, plus $1.

Rule in words: **Multiply by 3, then add 1.**

Rule using a variable: $3x + 1$.

**Example A**  
Make a **table of values** for the rule.  
**Multiply by 3, then add 1: $3x + 1$**

Evaluate the expression $3x + 1$ using 1, 2, 3, 4, and 5 for $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$3x$</th>
<th>$3x + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>16</td>
</tr>
</tbody>
</table>

For $x = 1, 3x + 1 = 3 \times 1 + 1 = 4.$

For $x = 2, 3x + 1 = 3 \times 2 + 1 = 7.$

For $x = 3, 3x + 1 = 3 \times 3 + 1 = 10.$

For $x = 4, 3x + 1 = 3 \times 4 + 1 = 13.$

For $x = 5, 3x + 1 = 3 \times 5 + 1 = 16.$

**How do you make a graph for a table of values?**

**Example C**  
Make a graph for the table of values in Example A.

Graph each of the ordered pairs in the table: (1, 4), (2, 7), (3, 10), (4, 13), (5, 16).

**Talk About It**

4. What do you notice about the points in the graph?

5. **Reasoning** How could you use the graph to find another ordered pair that fits the rule in Example A?

6. Do you think (6, 10) fits the rule? How can you tell without doing any computation?
Example 4-12  Explain why a telephone company would not set rates for telephone calls as depicted on the graph in Figure 4-19.

Solution  The graph does not depict a function. For example, a customer could be charged either $0.50 or $0.85 for a 2-min call; hence, not every input has a unique output. Also notice that the endpoints of each segment are shaded on both sides of each segment. This implies that for 3-, 4-, or 5-min calls there is no unique charge—another reason that the graph does not depict a function.

Suppose you join a DVD rental club where your cost per rental is $5. We have seen that one way to describe a function is by writing an equation. Based on the information in Table 4-3, the equation relating the number of videos rented to cost is \( C = n \cdot 5 \), or \( C = 5n \), where \( n \) is the number of DVDs rented.

### Table 4-3

<table>
<thead>
<tr>
<th>Number of DVDs Rented</th>
<th>Cost in Dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 \cdot 5 = 5</td>
</tr>
<tr>
<td>2</td>
<td>2 \cdot 5 = 10</td>
</tr>
<tr>
<td>3</td>
<td>3 \cdot 5 = 15</td>
</tr>
<tr>
<td>4</td>
<td>4 \cdot 5 = 20</td>
</tr>
<tr>
<td>5</td>
<td>5 \cdot 5 = 25</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( n )</td>
<td>( n \cdot 5 ) or ( 5n )</td>
</tr>
</tbody>
</table>
Algebraic Thinking

This could also be written as \( f(n) = 5n \), where \( f(n) \) is the cost of the rental in dollars. If we restrict the number of rentals to the first five natural numbers, the function can be described as the set of ordered pairs \( \{(1, 5), (2, 10), (3, 15), (4, 20), (5, 25)\} \). Figure 4-20 shows the graph of the function, which consists of five points that are not connected by a solid line. In graphing the function in Figure 4-20, we assume the domain to be the set \( \{1, 2, 3, 4, 5\} \). It does not make any sense to connect the points because one cannot, for example, rent 1.5 DVDs. However, to show that the points lie on a straight line, we connected them by a dashed line.

Sequences as Functions

Arithmetic, geometric, and other sequences introduced in Chapter 1 can be thought of as functions whose inputs are natural numbers and whose outputs are the terms of a particular sequence. For example, the arithmetic sequence 2, 4, 6, 8, \ldots, whose \( n \)th term is \( 2n \) can be described as a function from the set \( N \) (natural numbers) to the set \( E \) (even natural numbers) using the rule \( f(n) = 2n \), where \( n \) is a natural number.

Example 4-13

If \( f(n) \) denotes the \( n \)th term of a sequence, find \( f(n) \) in terms of \( n \) for each of the following:

a. An arithmetic sequence whose first term is 3 and whose difference is 3
b. A geometric sequence whose first term is 3 and whose ratio is 3
c. The sequence 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, \ldots

Solution  

a. The first term is 3, the second term is 3 + 3, or 2 \( \cdot \) 3, the third is 2 \( \cdot \) 3 + 3, or 3 \( \cdot \) 3, and the fourth term is 3 \( \cdot \) 3 + 3, or 4 \( \cdot \) 3, the \( n \)th term is \( n \cdot 3 \), and hence \( f(n) = n \cdot 3 = 3n \), where \( n \) is a natural number.

b. The first term is 3, the second \( 3 \cdot 3 \), or \( 3^2 \), the third \( 3 \cdot 3 \cdot 3 \), or \( 3^3 \), and so on. Hence, the \( n \)th term is \( 3^n \) and therefore \( f(n) = 3^n \), where \( n \) is a natural number.

c. The \( n \)th term is 1 + 2 + 3 + 4 + \ldots + \( n \). In Chapter 1 we saw that this sum equals \( \frac{n(n + 1)}{2} \), so the function is \( f(n) = \frac{n(n + 1)}{2} \), where \( n \) is a natural number.

NOW TRY THIS 4-9 Find the range of the function in Figure 4-20.
Composition of Functions

Consider the function machines in Figure 4-21. If 2 is entered in the top machine, then
\( f(2) = 2 + 4 = 6 \). The number 6 is then entered in the second machine and \( g(6) = 2 \cdot 6 = 12 \). The functions in Figure 4-21 illustrate the composition of two functions. In the composition of two functions, the range of the first function becomes the domain of the second function.

![Figure 4-21](image)

If the first function \( f \) is followed by a second function \( g \), as in Figure 4-21, we symbolize the composition of the functions as \( g \circ f \). If we input 3 in the function machines of Figure 4-21, then the output is symbolized by \( (g \circ f)(3) \). Because \( f \) acts first on 3, to compute \( (g \circ f)(3) \) we find \( f(3) = 3 + 4 = 7 \) and then \( g(7) = 2 \cdot 7 = 14 \). Hence, \( (g \circ f)(3) = 14 \). Notice that \( (g \circ f)(3) = g(f(3)) \). Also note that \( (g \circ f)(x) = g(f(x)) = 2 \cdot f(x) = 2(x + 4) \) and hence \( g(f(3)) = 2(3 + 4) = 14 \).

**Example 4-14**

If \( f(x) = 2x + 3 \) and \( g(x) = x - 3 \), find the following:

a. \( (f \circ g)(3) \)  
   b. \( (g \circ f)(3) \)  
   c. \( (f \circ g)(x) \)  
   d. \( (g \circ f)(x) \)

**Solution**

a. \( (f \circ g)(3) = f(g(3)) = f(3 - 3) = f(0) = 2 \cdot 0 + 3 = 3 \)

b. \( (g \circ f)(3) = g(f(3)) = g(2 \cdot 3 + 3) = g(9) = 9 - 3 = 6 \)

c. \( (f \circ g)(x) = f(g(x)) = 2 \cdot g(x) + 3 = 2(x - 3) + 3 = 2x - 6 + 3 = 2x - 3 \)

d. \( (g \circ f)(x) = g(f(x)) = f(x) - 3 = (2x + 3) - 3 = 2x \)

**Remark** Example 4-14 shows that composition of functions is not commutative, since \( (f \circ g)(3) \neq (g \circ f)(3) \).

We have seen that a function can be represented in a variety of ways. Pictures of sets with arrows and function machines are used mostly as pedagogical devices in learning the concept of a function. The most common representations are a table, an equation, or a graph. Depending on the situation, one representation may be more useful than another. For example, if the domain of a function has many elements, a table is not a convenient representation. Graphing calculators can be used to display a graph of most functions given by equations with specified domains.
Relations

As in the definition of a function from set $A$ to set $B$, a relation from set $A$ to set $B$ is a correspondence between elements of $A$ and elements of $B$, but unlike functions, we do not require that each element of $A$ be paired with one, and only one, element of $B$. Consequently, any set of ordered pairs is a relation. Notice that every function is a relation, but not every relation is a function. Examples of relations include the following:

- “is a daughter of”  
- “is the same color as”  
- “is less than”  
- “is greater than or equal to”

Consider the relation “is a sister of.” Figure 4-23 illustrates this relation among children on a playground, with letters $A$ through $J$ representing the childrens’ names. An arrow from $I$ to $J$ indicates that $I$ “is a sister of” $J$. Notice the arrows from $F$ to $G$ and from $G$ to $F$, which indicate that $F$ is a sister of $G$ and $G$ is a sister of $F$. This implies that $F$ and $G$ are girls. On the other hand, the absence of an arrow from $J$ to $I$ implies that $J$ is not a sister of $I$. Thus, $I$ is a girl and $J$ is a boy.

Figure 4-23
Another way to show the relation “is a sister of” is to write the relation “A is a sister of B” as an ordered pair \((A, B)\). Notice that \((B, A)\) means that B is a sister of A. Using this notation, the relation “is a sister of” can be described for the children on the playground as the set
\[
\{(A, B), (A, C), (A, D), (C, A), (C, B), (C, D), (D, A), (D, B), (D, C), (F, G), (G, F), (I, J)\}
\]
Observe that this set is a subset of \(\{A, B, C, D, E, F, G, H, I, J\} \times \{A, B, C, D, E, F, G, H, I, J\}\). This observation motivates the following definition of a relation.

**Definition**

A relation from set \(A\) to set \(B\) is a subset of \(A \times B\); that is, \(R\) is a relation from set \(A\) to set \(B\) if, and only if, \(R \subseteq A \times B\).

In the definition, the phrase “from \(A\) to \(B\)” means that the first components in the ordered pairs are elements of \(A\) and the second components are elements of \(B\). If \(A = B\), we say that the relation is on \(A\).

**Properties of Relations**

Figure 4-24 represents a set of children in a small group. The children have drawn all possible arrows representing the relation “has the same first letter in his or her name as.” Notice that the children were very careful to observe that each child in the group has the same first initial as himself or herself. Three properties of relations are illustrated in Figure 4-24.

**Definition of the Reflexive Property**

A relation \(R\) on a set \(X\) is reflexive if, and only if, for every element \(a \in X\), \(a\) is related to \(a\). That is, for every \(a \in X\), \((a, a) \in R\).
In the diagram, there is a loop at every point. For example, Rick has the same first initial as himself, namely R. A relation such as “is taller than” is not reflexive because people cannot be taller than themselves.

**Definition of the Symmetric Property**

A relation $R$ on a set $X$ is symmetric if, and only if, for all elements $a$ and $b$ in $X$, whenever $a$ is related to $b$, then $b$ also is related to $a$. That is, if $(a, b) \in R$, then $(b, a) \in R$.

In terms of the diagram, every pair of points that has an arrow headed in one direction also has a return arrow. For example, if Bill has the same first initial as Betty, then Betty has the same first initial as Bill. A relation such as “is a brother of” is not symmetric since Dick can be a brother of Jane, but Jane cannot be a brother of Dick.

**Definition of the Transitive Property**

A relation $R$ on a set $X$ is transitive if, and only if, for all elements $a$, $b$, and $c$ of $X$, whenever $a$ is related to $b$ and $b$ is related to $c$, then $a$ is related to $c$. That is, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

**Remark** $a$, $b$, and $c$ do not have to be different. Three symbols are used to allow for difference.

Notice that the relation in Figure 4-24 is transitive. For example, if Carol has the same first initial as Candy, and Candy has the same first initial as Cathy, then Carol has the same first initial as Cathy. A relation such as “is the father of” is not transitive since, if Tom Jones is the father of Tom Jones, Jr. and Tom Jones, Jr. is the father of Joe Jones, then Tom Jones is not the father of Joe Jones. He is, instead, the grandfather of Joe Jones.

The relation “is the same color as” is reflexive, symmetric, and transitive. The common relation “is equal to” also satisfies all three properties. In general, relations that satisfy all three properties are called **equivalence relations**.

**Definition**

An *equivalence relation* is any relation $R$ that satisfies the reflexive, symmetric, and transitive properties.

The most natural equivalence relation encountered in elementary school is “is equal to” on the set of all numbers. In subsequent chapters, we will see more examples of equivalence relations.

The symmetric property of a relation is particularly useful in determining a symmetric nature of functions. Consider the relation $x + y = 10$, where $x$ and $y$ are whole numbers. The

* Using the terminology of Chapter 14, a relation is symmetric if the line $y = x$ is the line of symmetry of its graph. However, a graph of a relation can have other lines of symmetry.
relation consists of the 11 ordered pairs graphed in Figure 4-25; the points lie on the dotted straight line. Notice that if the pair \((a, b)\) is in the relation—that is, is on the graph—so is the pair \((b, a)\). For example, \((1, 9)\) is on the graph because \(1 + 9 = 10\), but then \((9, 1)\) is also on the graph because \(9 + 1 = 10\). Also notice that the relation is a function. This can be seen from the graph, where for each input \(x, x = 0, 1, 2, 3, \ldots, 10\), there is exactly one output \(y\). We could also see that the relation is a function because \(x + y = 10\) gives \(y = 10 - x\); that is, for each \(x\) in the domain, there is a unique \(y\). The domain and range of the function are the same set \(\{0, 1, 2, 3, \ldots, 10\}\).

![Figure 4-25](image)

**NOW TRY THIS 4-11**

a. Explain why the domain and the range of the function \(y = 10 - x\), where both \(x\) and \(y\) are whole numbers, is the set \(\{0, 1, 2, 3, \ldots, 10\}\).

b. Show why the function \(y = x + 10\), where \(x\) and \(y\) are whole numbers, is not a symmetric relation.

---

**Assessment 4-3A**

1. The following sets of ordered pairs are functions. Give a rule that could describe each function.
   
   a. \(\{(2, 4), (3, 6), (9, 18), (12, 24)\}\)
   
   b. \(\{(2, 8), (5, 11), (7, 13), (4, 10)\}\)

2. Which of the following are functions from the set \(\{1, 2, 3\}\) to the set \(\{a, b, c, d\}\)? If the set of ordered pairs is not a function, explain why not.
   
   a. \(\{(1, a), (2, b), (3, c), (1, d)\}\)
   
   b. \(\{(1, a), (2, b), (3, a)\}\)

3. a. Draw an arrow diagram of a function with domain \(\{1, 2, 3, 4, 5\}\) and range \(\{a, b\}\).
   
   b. How many possible functions are there in part (a)?

4. Suppose \(f(x) = 2x + 1\) and the domain is \(\{0, 1, 2, 3, 4\}\). Describe the function in the following ways:
   
   a. Draw an arrow diagram involving two sets.
   
   b. Use ordered pairs.
   
   c. Make a table.
   
   d. Draw a graph to depict the function.
5. Determine which of the following are functions from \( W = \{0, 1, 2, 3, \ldots \} \) or a subset of \( W \) to \( W \). If your answer is that it is not a function, explain why not.
   a. \( f(x) = 2 \) for all \( x \in W \)
   b. \( f(x) = x \)
   c. \( f(x) \) is the sum of the digits in \( x \) for all \( x \in W \).

6. a. Make an arrow diagram for each of the following:
   (i) Rule: “when doubled is”
   
   ![Diagram A](image)
   ![Diagram B](image)
   (ii) Rule: “is greater than”
   
   ![Diagram A](image)
   ![Diagram B](image)

   b. Which, if any, of the parts in (a) exhibits a function from \( A \) to \( B \)? If it is a function, tell why and find the range of the function.

7. The dosage of a certain drug is related to the weight of a child as follows: 50 mg of the drug and an additional 15 mg for each 2 lb or fraction of 2 lb of body weight above 30 lb. Sketch the graph of the dosage as a function of the weight of a child for children who weigh between 20 and 40 lb.

8. If taxi fares are $3.50 for the first half mile and $0.75 for each additional quarter mile, answer the following:
   a. What is the fare for a 2-mi trip?
   b. Write a rule for computing the fare for an \( n \)-mile trip by taxi if \( n \) is a natural number.

9. For each of the following, guess what might be Latifah’s rule. In each case, if \( n \) is your input and \( L(n) \) is Latifah’s answer, express \( L(n) \) in terms of \( n \).
   a. \( \begin{array}{c|c|c} \text{You} & \text{Latifah} & \text{You} \text{ Latifah} \\ \hline 3 & 8 & 0 \\ 4 & 11 & 3 \\ 5 & 14 & 5 \\ 10 & 29 & 8 \end{array} \)
   b. \( \begin{array}{c|c|c} \text{You} & \text{Latifah} \\ \hline 3 & 8 \\ 4 & 11 \\ 5 & 14 \\ 10 & 29 \end{array} \)

10. In the *Principles and Standards* for grades 6–8, the “Algebra” section (p. 229) poses the following problem. Quick-Talk advertises monthly cellular phone service for $0.50 a minute for the first 60 min but only $0.10 a minute for each minute thereafter. Quick-Talk charges for the exact amount of time used. Answer the following:
   a. Make one graph showing the cost per minute as a function of number of minutes and the other showing the total cost for calls as a function of the number of minutes up to 100 min.
   b. If you connect the points in the second graph in part (a), what kind of assumption needs to be made about the way the telephone company charges phone calls?
   c. Why does the total cost for calls consist of two line segments? Why is one part steeper than the other?
   d. The function representing the total cost for calls as a function of number of minutes talked can be represented by two equations. Write these equations.

11. For each of the following sequences, find a possible function \( f(n) \) whose domain is the set of natural numbers and whose outputs are the terms of the sequence.
   a. 3, 8, 13, 18, 23, \ldots
   b. 3, 9, 27, 81, 243, \ldots

12. Consider the following two function machines. Find the final output for each of the following inputs:
   a. 5
   b. 10

13. Let \( t(n) \) represent the \( n \)th term of a sequence for \( n \in N \). Answer the following:
   a. If \( t(n) = 4n - 3 \), determine which of the following are output values of the function:
      (i) 1  (ii) 385  (iii) 389  (iv) 392
   b. If \( t(n) = n^2 \), determine which of the following are output values of the function:
      (i) 0  (ii) 25  (iii) 625  (iv) 1000  (v) 90
   c. If \( t(n) = n(n - 1) \), determine which of the following are in the range of the function:
      (i) 0  (ii) 20  (iii) 999

14. Consider a function machine that accepts inputs as ordered pairs. Suppose the components of the ordered pairs are natural numbers and the first component is the length of a rectangle and the second is its width. The following machine computes the perimeter (the distance around a figure) of the rectangle. Thus, for a
rectangle whose length, \( l \), is 3 and whose width, \( w \), is 2, the input is (3, 2) and the output is \( 2 \cdot 3 + 2 \cdot 2 \), or 10. Answer each of the following:

a. For each of the following inputs, find the corresponding output: (1, 7), (2, 6), (6, 2), (5, 5).

b. Find the set of all the inputs for which the output is 20.

c. What is the domain and the range of the function?

15. The following graph shows the relationship between the number of cars on a certain road and the time of day for times between 5:00 A.M. and 9:00 A.M.: 

a. What was the increase in the number of cars on the road between 6:30 A.M. and 7:00 A.M.?

b. During which half hour was the increase in the number of cars the greatest?

c. What was the increase in the number of cars between 8:00 A.M. and 8:30 A.M.?

d. During which half hour(s) did the number of cars decrease? By how much?

e. The graph for this problem is composed of segments rather than just points as in Figure 4-20. Why do you think segments are used here instead of just points?

16. A ball is shot out of a cannon at ground level. We know that its height \( H \) in feet after \( t \) sec is given by the function \( H(t) = 128t - 16t^2 \).

a. Find \( H(2) \), \( H(6) \), \( H(3) \), and \( H(5) \). Why are some of the outputs equal?

b. Graph the function and from the graph find at what instant the ball is at its highest point. What is its height at that instant?

c. How long will it take the ball to hit the ground?

d. What is the domain of \( H \)?

e. What is the range of \( H \)?

17. For each of the following sequences of matchstick figures, let \( S(n) \) be the function giving the total number of matchsticks in the \( n \)th figure.

a. For each of the following, find the total number of matchsticks in the fourth figure.

b. For each of the following, find as simple a formula as possible for \( S(n) \) in terms of \( n \).

\[(i)\]  \[S_1 = 1, S_2 = 5, S_3 = 9, S_4 = 13, \ldots\]

\[(ii)\]  \[S_1 = 3, S_2 = 7, S_3 = 11, S_4 = 15, \ldots\]

18. Assume the pattern continues for each of the following sequences of square tile figures. Let \( S(n) \) be the function giving the total number of tiles in the \( n \)th figure. For each of the following, find a formula for \( S(n) \) in terms of \( n \). In part (b), each square is divided into four squares in the subsequent figure.

a. 

b. 

19. A function can be represented as a set of ordered pairs where the set of all the first components is the domain and where the set of all the second components is the range. Is the converse also true? That is, is every set of ordered pairs a function whose domain is the set of first components and whose range is the set of second components? Justify your answer.

20. Which of the following equations or inequalities represent functions and which do not? In each case \( x \) and \( y \) are whole numbers. Justify your answers.

a. \( x + y = 2 \)

b. \( x - y < 2 \)

c. \( y = x^3 + x \)

d. \( xy = 2 \)
21. Which of the following are graphs of functions and which are not? Justify your answers.

![Graphs](image)

22. Suppose each point in the figure represents a child on a playground, the letters represent their names, and an arrow going from I to J means that I "is the sister of" J.

![Diagram](image)

a. Based on the information in the figure, who are definitely girls and who are definitely boys?

b. Suppose we write "A is the sister of B" as an ordered pair \((A, B)\). Based on the information in the diagram, write the set of all such ordered pairs.

c. Is the set of all ordered pairs in (b) a function with the domain equal to the set of all first components of the ordered pairs and with the range equal to the set of all second components?

---

**Assessment 4-3B**

1. The following sets of ordered pairs are functions. Give a rule that could describe each function.
   a. \(\{(5,3),(7,5),(11,9),(14,12)\}\)
   b. \(\{(2,5),(3,10),(4,17),(5,26)\}\)

2. Which of the following are functions from the set \(\{1,2,3\}\) to the set \(\{a,b,c,d\}\)? If the set of ordered pairs is not a function, explain why not.
   a. \(\{(1,c),(3,d)\}\)
   b. \(\{(1,a),(1,b),(1,c)\}\)

3. a. Draw an arrow diagram of a function with domain \(\{1,2,3\}\) and range \(\{a,b\}\).
   b. How many possible functions are there in part (a)?

4. Suppose \(f(x) = 2(x + 1)\) and the domain is \(\{0,1,2,3,4\}\). Describe the function in the following ways:
   a. Draw an arrow diagram involving two sets.
   b. Use ordered pairs.
   c. Make a table.
   d. Draw a graph to depict the function.

5. Determine which of the following are functions from \(W = \{0,1,2,3,\ldots\}\) or a subset of \(W\) to \(W\). If your answer is that it is not a function, explain why not.
   a. \(f(x) = 0\) if \(x \in \{0,1,2,3\}\), and \(f(x) = 3\) if \(x \in \{0,1,2,3\}\)
   b. \(f(x) = 0\) for all \(x \in W\) and \(f(x) = 1\) if \(x \in \{3,4,5,6,\ldots\}\)
   c. \(f(x)\) is the unit digit in \(x\) for all \(x \in W\).

6. Given the following arrow diagrams for functions from \(A\) to \(B\), give a possible rule for the function:
   a. 
   b. 

7. According to wildlife experts, the rate at which crickets chirp is a function of the temperature; specifically, \(C = T - 40\), where \(C\) is the number of chirps every 15 sec and \(T\) is the temperature in degrees Fahrenheit.
   a. How many chirps does the cricket make per second if the temperature is 70°F?
   b. What is the temperature if the cricket chirps 40 times in 1 min?
8. For each of the following, guess what might be Latifah's rule. In each case, if \( n \) is your input and \( L(n) \) is Latifah's answer, express \( L(n) \) in terms of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>You</th>
<th>Latifah</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td>42</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>72</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

b. | \( n \) | You | Latifah |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>32</td>
</tr>
<tr>
<td>64</td>
<td></td>
<td>64</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>1024</td>
</tr>
</tbody>
</table>

9. The Principles and Standards for grades 6–8 points out that “in their study of algebra, middle-grades students should encounter questions that focus on quantities that change” (p. 229). It poses the following problem.

ChitChat charges \$0.45 a minute for cellular phone calls. The cost per minute does not change, but the total cost changes as the telephone is used.

Cellular Phone Costs per Minute

<table>
<thead>
<tr>
<th>Number of minutes</th>
<th>Cost per minute</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0.45</td>
</tr>
<tr>
<td>1</td>
<td>$0.90</td>
</tr>
<tr>
<td>2</td>
<td>$1.35</td>
</tr>
<tr>
<td>3</td>
<td>$1.80</td>
</tr>
<tr>
<td>4</td>
<td>$2.25</td>
</tr>
<tr>
<td>5</td>
<td>$2.70</td>
</tr>
<tr>
<td>6</td>
<td>$3.15</td>
</tr>
<tr>
<td>7</td>
<td>$3.60</td>
</tr>
</tbody>
</table>

Total Cellular Phone Costs

<table>
<thead>
<tr>
<th>Number of minutes</th>
<th>Total cost for calls</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0.45</td>
</tr>
<tr>
<td>1</td>
<td>$0.90</td>
</tr>
<tr>
<td>2</td>
<td>$1.35</td>
</tr>
<tr>
<td>3</td>
<td>$1.80</td>
</tr>
<tr>
<td>4</td>
<td>$2.25</td>
</tr>
<tr>
<td>5</td>
<td>$2.70</td>
</tr>
<tr>
<td>6</td>
<td>$3.15</td>
</tr>
<tr>
<td>7</td>
<td>$3.60</td>
</tr>
</tbody>
</table>

10. For each of the following sequences, discover a possible pattern and find a function whose domain is the set of natural numbers and whose outputs are the terms of the sequence:

a. 2, 4, 6, 8, 10, ...
b. 1, 3, 9, 27, 81, ...
c. 2, 2 + 4, 2 + 4 + 6, 2 + 4 + 6 + 8, ...

11. Consider two function machines that are placed as shown. Find the final output for each of the following inputs:

a. 5
b. 3
c. 10
d. \( a \)

12. Let \( t(n) \) represent the \( n \)th term of a sequence for \( n \in \mathbb{N} \). Answer the following:

a. If \( t(n) = n^2 \), determine which of the following are output values of the function:
   (i) 1 (ii) 4 (iii) 9 (iv) 10 (v) 900
b. If \( t(n) = n(n + 1) \), determine which of the following are in the range of the function:
   (i) 2 (ii) 12 (iii) 2550 (iv) 2600

13. Consider a function machine that accepts inputs as ordered pairs. Suppose the components of the ordered pairs are natural numbers and the first component is the length of a rectangle and the second is its width. The following machine computes the perimeter (the distance around a figure) of the rectangle. Thus, for a rectangle whose length, \( l \), is 3 and whose width, \( w \), is 1, the input is (3, 1) and the output is \( 2 \cdot 3 + 2 \cdot 1 \), or 8. Answer each of the following:

a. For each of the following inputs, find the corresponding output: (1, 4), (2, 1), (1, 2), (2, 2), (x, y).
b. Find the set of all the inputs for which the output is 20.
c. Is (2, 2) a possible output? Explain.

14. A health club charges a one-time initiation fee of \$100 plus a membership fee of \$40 per month.
a. Write an expression for the cost function \( C(x) \) that gives the total cost for membership at the health club for \( x \) months.

b. Draw the graph of the function in (a).

c. The health club decided to give its members an option of a higher initiation fee but a lower monthly membership charge. If the initiation fee is $300 and the monthly membership fee is $30, use a different color and draw on the same set of axes the cost graph under this plan.

d. Determine after how many months the second plan is less expensive for the member.

15. A ball is shot straight up at ground level. We know that its height \( H \) in feet after \( t \) sec is given by the function \( H(t) = 128t - 16t^2 \).

a. Graph the function and from the graph find at what instant the ball is at its highest point. What is its height at that instant?

b. Find from the graph all \( t \) such that \( H(t) = H(1) \).

c. How long will it take the ball to hit the ground? Check your answer.

d. What is the domain of \( H \)?

e. What is the range of \( H \)?

16. A rectangular plot is bounded on one side by a straight river and on the other sides by a fence. Suppose 900 yd of fence are available and the length of the side of the rectangle parallel to the river is denoted by \( x \).

a. Find an expression for the area \( A(x) \) in terms of \( x \).

b. Graph \( A(x) \).

c. Use the graph in (b) or your calculator to estimate the length and width of the rectangle for which the area will be the largest.

17. For each of the following sequences of matchstick figures, assume that your discovered pattern continues and let \( S(n) \) be the function giving the total number of matchsticks in the \( n \)th figure.

a. For each of the following figures assume that your pattern continues and find the total number of matchsticks in the fourth figure.

b. For each of the following, find a formula for \( S(n) \) in terms of \( n \).

(i) 

(ii) 

18. You are 20 mi away from your home and start driving away from home at a constant speed of 60 mph. Write the distance \( S \) from home as a function of \( t \), the time in hours of driving.

19. Assume the pattern continues for each of the following sequences of square tile figures. Let \( S(n) \) be the function giving the total number of tiles in the \( n \)th figure. For each of the following, find a formula for \( S(n) \) in terms of \( n \).

a. 

b. 

c. 

d. 

20. A function can be represented as a set of ordered pairs where the set of all the first components is the domain and the set of all the second components is the range. If each ordered pair \((a,b)\) is replaced by \((b,a)\), is the new set still a function?

21. Which of the following equations or inequalities represent functions and which do not? In each case \( x \) and \( y \) are whole numbers. Justify your answers.

a. \( x - y = 2 \)

b. \( x + y < 20 \)

c. \( y = 2x^2 \)

d. \( y = x^3 - 1 \)

22. Which of the following are graphs of functions and which are not? Justify your answers.

a. 

b. 

c. 

d. 

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23. a. Which of the following relations from the set $W$ of whole numbers to $W$ have the symmetric property? Justify your answers.

(i) $x + y = 10$  
(ii) $x - y = 100$  
(iii) $xy = 100$  
(iv) $y = x$

b. Which of the relations in part (a) are functions? Justify your answer.

24. Suppose each point in the figure represents a child on a playground, the letters represent their names, and an arrow going from $I$ to $J$ means that $I$ “is the sister of” $J$.

a. Based on the information in the figure, who are definitely girls and who are definitely boys?

b. Suppose we write “$A$ is the sister of $B$” as an ordered pair $(A, B)$. Based on the information in the diagram, write the set of all such ordered pairs.

c. Is the set of all ordered pairs in (b) a function with the domain equal to the set of first components of the ordered pairs to the set of second components?

25. Which of the following are functions and which are relations but not functions from the set of first components of the ordered pairs to the set of second components?

a. $\{(\text{Montana, Helena}), (\text{Oregon, Salem}), (\text{Illinois, Springfield}), (\text{Arkansas, Little Rock})\}$

b. $\{(\text{Pennsylvania, Philadelphia}), (\text{New York, Albany}), (\text{New York, Niagara Falls}), (\text{Florida, Ft. Lauderdale})\}$

c. $\{(x, y) \mid x \text{ resides in Birmingham, Alabama, and } x \text{ is the mother of } y, \text{ where } y \text{ is a U.S. resident}\}$

d. $\{(1,1), (2,4), (3,9), (4,16)\}$

e. $\{(x, y) \mid x \text{ and } y \text{ are natural numbers and } x + y \text{ is an even number}\}$

26. a. Consider the relation consisting of ordered pairs $(x, y)$ such that $y$ is the biological mother of $x$. Is this a function whose domain is the set of all people?

b. Like in part (a) but now $y$ is a brother of $x$. Is the relation a function from the set of all boys to the set of all boys?

27. Tell whether each of the following is reflexive, symmetric, or transitive on the set of all people. Which are equivalence relations?

a. “Is a parent of”

b. “Is the same age as”

c. “Has the same last name as”

d. “Is the same height as”

e. “Is married to”

f. “Lives within 10 mi of”

g. “Is older than”

28. Tell whether each of the following is reflexive, symmetric, or transitive on the set of subsets of a nonempty set. Which are equivalence relations?

a. “Is equal to”

b. “Is a proper subset of”

c. “Is not equal to”

d. “Has the same cardinal number as”
complement of \(A\). Notice that the input in this function is a subset of \(S\) and the output is a subset of \(S\). Answer the following:

a. Explain why \(f\) is a function and describe the domain and the range of \(f\).

b. If there are 20 children in the class, what are the number of elements in the domain and the number in the range? Explain.

c. Is the function in this question a one-to-one correspondence? Justify your answer.

Open-Ended

5. Examine several newspapers and magazines and describe at least three examples of functions that you find. What is the domain and range of each function?

6. Give at least three examples of functions from \(A\) to \(B\) where neither \(A\) nor \(B\) is a set of numbers.

7. Draw a sequence of matchstick figures and describe the pattern in words. Find as simple an expression as possible for \(S(n)\), the total number of matchsticks in the \(n\)th figure.

8. A function whose output is always the same regardless of the input is a constant function. Give several examples of constant functions from real life.

9. A function whose output is the same as its input is an identity function. Give several concrete examples of identity functions.

Cooperative Learning

10. Each person in a group picks a natural number and uses it as an input in the following function machine:

   ![Function Machine Diagram]

   a. Compare your answers. Based on the answers, make a conjecture about the range of the function.

   b. Based on your answer in (a), graph the function.

   c. Write the function in the simplest possible way using \(f(x)\) notation.

   d. Justify your conjecture in (a).

   e. Make up similar function machines and try different inputs in your group.

   f. Devise a function machine in which the machine performs several operations but the output is always the same as the input. Exchange your answer with someone in the group and check that the other person’s function machine performs as required.

11. In a group of four, work through the following. You will need a metric tape or meterstick.

   a. Place your math book on a desk and measure the distance (to the nearest centimeter) from the floor to the top of the book. Record the distance.

   b. Place a second math book on top of the first and measure the distance (to the nearest centimeter) from the floor to the top of the second book. Record the distance.

   c. Continue this procedure for all four of your math books and complete the following table and graph:

<table>
<thead>
<tr>
<th>Number of books</th>
<th>Distance from floor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>70</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
</tr>
<tr>
<td>4</td>
<td>90</td>
</tr>
</tbody>
</table>

   d. Without measuring, what is the distance from the floor with 0 books? 5 books?

   e. Write a rule or function for \(d(x)\), where \(d(x)\) is the distance above the floor to the top of the stack of books and \(x\) is the number of books.

   f. Suppose the distance from the floor to the ceiling is 2.5 m. If you stack the books as described above, how many books would be needed to reach the ceiling?

   g. The function \(b(x) = 3x + 70\) represents the height of another stack of \(x\) math books (in centimeters) on a cabinet. What does the function tell you about the height of the cabinet? What does it tell you about the thickness of each book?

   h. Suppose that a table with a stack of similar math books (more than 10) is 200 cm high. If the top math book is removed, the height is 197 cm. If a second book is removed, the height is 194 cm. What is the height if 5 books are removed?

   i. Write a function \(b(x)\) for the height of the stack in part (h) after \(x\) books are removed.

Questions from the Classroom

12. A student claims that the following machine does not represent a function machine because it accepts two inputs at once rather than a single input. How do you respond?
13. A student asks, “If every sequence is a function, is it also true that every function is a sequence?” How do you respond?

14. A student claims that the following does not represent a function, since all the values of \( x \) correspond to the same number.

\[
\begin{array}{c|cccccc}
  x & 0 & 1 & 2 & 3 & 4 & 5 \\
  y & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

How do you respond?

15. A student thinks that the function \( f(x) = 3x + 5 \) with domain of all whole numbers is a one-to-one correspondence and he would like to know why. How do you respond?

16. A student wants to know why sometimes it is incorrect to connect points on the graph of a function. How do you respond?

Review Problems

17. Solve the following equations, if possible:
   a. \( 3x - 1 = x + 99 \)
   b. \( 2(5x + 1) - 11 = x + 9 \)
   c. \( 3(x - 1) = 2(x - 1) + 99 \)
   d. \( 5(2x - 6) = 3(2x - 6) \)

18. Solve the following problem by setting up an appropriate equation:
   Two cars, each traveling at a constant speed—one 60 mph and the other 70 mph—start at the same time from the same point traveling in the same direction. After how many hours will the distance between them be 40 mi?

National Assessment of Educational Progress (NAEP) Questions

<table>
<thead>
<tr>
<th>In</th>
<th>Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>15</td>
<td>31</td>
</tr>
<tr>
<td>38</td>
<td></td>
</tr>
</tbody>
</table>

The table shows how the “In” numbers are related to the “Out” numbers. When 38 goes in, what number comes out?

a. 41  b. 51  c. 54  d. 77

NAEP, Grade 4, 2007

Each figure in the pattern below is made of hexagons that measure 1 centimeter on each side.

Figure 1
Perimeter = 6 cm

Figure 2
Perimeter = 10 cm

Figure 3
Perimeter = 14 cm

Figure 4
Perimeter = 18 cm

If the pattern of adding one hexagon to each figure is continued, what will be the perimeter of the 25th figure in the pattern? Show how you got your answer.

NAEP, Grade 8, 2007

In the equation \( y = 4x \), if the value of \( x \) is increased by 2, what is the effect on the value of \( y \)?

a. It is 8 more than the original amount.
b. It is 6 more than the original amount.
c. It is 2 more than the original amount.
d. It is 16 times the original amount.
e. It is 8 times the original amount.

NAEP, Grade 8, 2007

Third International Mathematics and Science Study (TIMSS) Questions

A number machine takes a number and operates on it. When the Input Number is 5, the Output Number is 9, as shown below.

\[
\begin{array}{c}
5 \times 2 \quad + 2 \quad 12 \quad - 3 \quad 9 \\
\end{array}
\]

When the Input Number is 7, which of these is the Output Number?

a. 11  b. 13  c. 14  d. 25

TIMSS 2003, Grade 4
Ali had 50 apples. He sold some and then had 20 left. Which of these is a number sentence that shows this?

- a. □ - 20 = 50
- b. 20 - □ = 50
- c. □ - 50 = 20
- d. 50 - □ = 20

TIMSS 2003, Grade 4

The objects on the scale make it balance exactly. On the left pan there is a 1 kg weight (mass) and half a brick. On the right pan there is one brick.

What is the weight (mass) of one brick?

- a. 0.5 kg
- b. 1 kg
- c. 2 kg
- d. 3 kg

TIMSS 2003, Grade 8

**Hint for Solving the Preliminary Problem**

Call each student number $x$ and the final answer $a$. Write an equation involving $x$ and $a$ and solve for $x$ in terms of $a$.

**Chapter Outline**

I. Variables
- A. Unknown in an equation
- B. Changing quantity
- C. To apply algebra in solving problems
- D. In a spreadsheet

II. Equations
- A. Addition property: For numbers $a$, $b$, and $c$, if $a = b$, then $a + c = b + c$.
- B. Multiplication property: For numbers $a$, $b$, and $c$, if $a = b$, then $ac = bc$.
- C. Cancellation properties: For numbers $a$, $b$, and $c$,
  1. if $a + c = b + c$, then $a = b$.
  2. if $c \neq 0$, and $ac = bc$, then $a = b$.
- D. Equality is not affected if we substitute a number for its equal.
- E. Addition and subtraction property of equations
  1. If $a = b$ and $c = d$, then $a + c = b + d$ and $a - c = b - d$.
- F. The distributive properties
  1. $a(b + c) = ab + ac$
  2. $a(b - c) = ab - ac$
- G. Solving equations
- H. Application problems

III. Functions and Relations
- A. A function from set $A$ to $B$ is a correspondence in which each element $a \in A$ is paired with one, and only one, element $b \in B$. If the function is denoted by $f$, we write $f(a) = b$. The element $a \in A$ is the input and $f(a)$ is the output. $A$ is the domain of the function. $B$ is any set containing all the outputs. The set of all the outputs is the range of the function.
- B. A function can be represented by a rule, a table, an equation, an arrow diagram, a function machine, a set of ordered pairs, or a graph.
- C. A sequence is a function whose domain is $\mathbb{N}$, the set of natural numbers.
- D. Any set of ordered pairs is a relation.
  1. A relation from $A$ to $B$ is a subset of $A \times B$.
  2. A relation may have one or more of the following properties:
     a. Reflexive
     b. Symmetric
     c. Transitive
1. There are 13 times as many students as professors at a college. Use \( S \) for the number of students and \( P \) for the number of professors to represent the given information.

2. Write a sentence that gives the same information as the following equation: \( A = 103B \), where \( A \) is the number of girls in a neighborhood and \( B \) is the number of boys.

3. Write an equation to find the number of feet given the number of yards (let \( f \) be the number of feet and \( y \) be the number of yards).

4. The sum of a set of \( n \) whole numbers is \( S \). If each number is multiplied by 10 and then decreased by 10, what is the sum of the new set in terms of \( n \) and \( S \)?

5. I am thinking of a whole number. If I divide it by 5, write an equation to find the number of feet given the number of yards.

6. a. Think of a number.
   Add 17.
   Double the result.
   Subtract 4.
   Double the result.
   Add 20.
   Divide by 4.
   Subtract 20.
   Your answer will be your original number. Explain how this trick works.

   b. Fill in three more steps that will take you back to your original number.

   c. Make up a series of instructions such that you will always get back to your original number.

7. Find all the values of \( x \) that satisfy the following equations:
   a. \( 4x - 2 = 3x + 10 \)
   b. \( 4(x - 12) = 2x + 10 \)
   c. \( 4(7x - 21) = 14(7x - 21) \)
   d. \( 2(3x + 5) = 6x + 11 \)
   e. \( 3(x + 1) + 1 = 3x + 4 \)

8. Mike has 3 times as many baseball cards as Jordan, who has twice as many cards as Paige. Together, the three children have 999 cards. Set up an equation in one variable and find how many cards each child has.

9. Jeannie has 10 books overdue at the library. She remembers she checked out 2 science books two weeks before she checked out 8 children’s books. The daily fine per book is $0.20. If her total fine was $11.60, how long was each book overdue?

10. Three children deliver all the newspapers in a small town. Jacobo delivers twice as many papers as Dahlia, who delivers 100 more papers than Rashid. If altogether 500 papers are delivered, how many papers does each child deliver?

11. Which of the following sets of ordered pairs are functions from the set of first components to the set of second components?
   a. \( \{(a,b),(c,d),(e,a),(f,g)\} \)
   b. \( \{(a,b),(a,e),(b,b),(b,c)\} \)
   c. \( \{(a,b),(b,a)\} \)

12. Given the following function rules and the domains, find the associated ranges:
   a. \( f(x) = x + 3; \text{domain} = \{0,1,2,3\} \)
   b. \( f(x) = 3x - 1; \text{domain} = \{5,10,15,20\} \)
   c. \( f(x) = x^2; \text{domain} = \{0,1,2,3,4\} \)
   d. \( f(x) = x^2 + 3x + 5; \text{domain} = \{0,1,2\} \)

13. Which of the following correspondences from \( A \) to \( B \) describe a function? If a correspondence is a function, find its range. Justify your answers.
   a. \( A \) is the set of college students, and \( B \) is the set of majors. To each college student corresponds his or her major.
   b. \( A \) is the set of books in the library, and \( B \) is the set of natural numbers. To each book corresponds the number of pages in the book.
   c. \( A = \{(a,b) | a \in N \text{ and } b \in N\}, \text{and } B = N. \) To each element of \( A \) corresponds the number \( 4a + 2b. \)
   d. \( A = N \text{ and } B = N. \) If \( x \) is even, then \( f(x) = 0, \) and if \( x \) is odd, then \( f(x) = 1. \)
   e. \( A = N \text{ and } B = N. \) To each natural number corresponds the sum of its digits.

14. A health club charges an initiation fee of $200, which gives 1 month of free membership, and then charges $55 per month.
   a. If \( C(x) \) is the total cost of membership in the club for \( x \) months, express \( C(x) \) in terms of \( x. \)
   b. Graph \( C(x) \) for the first 12 months.
   c. Use the graph in (b) to find when the total cost of membership in the club will exceed $600.
   d. When will the total cost of membership exceed $600?

15. If the rule for the function is \( f(x) = 4x - 5 \) and \( f(x) = 15 \) is the output, what is the input?
16. Which of the following graphs represent functions? Tell why.

a.

b.

c.

17. a. Jilly is building towers with cubes, placing one cube on top of another and painting the tower (including the top and the bottom, but not the faces touching each other). Find the number of square faces that Jilly needs to paint for towers made of 1, 2, 3, 4, 5, and 6 cubes by filling in the following table:

<table>
<thead>
<tr>
<th># of Cubes</th>
<th># of Squares to Paint</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
</tr>
<tr>
<td>6</td>
<td>26</td>
</tr>
</tbody>
</table>

b. Graph the information you found in part (a) where the number of cubes in a tower is on the horizontal x-axis and the number of squares to be painted is on the vertical axis.

c. If \( x \) is the number of cubes in a tower and \( y \) is the corresponding number of squares to be painted, write an equation that gives \( y \) as a function of \( x \).

d. Is the graph describing the number of squares as a function of the number of cubes used a straight line?

Selected Bibliography


